Major Invariants of Zero-Divisor Graphs of Z_n: A Unified Prime-Factorization Framework

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Abstract

In the case of the classical zero-divisor graph Γ(Zn), it is known that it is always connected when n is composite and the diameter is at most three; formulas for the major invariants are typically given in individual case studies. This paper provides a common uniform prime-factorization framework for the order, size, degree sequence, clique number, chromatic number, ordinary domination, connected domination, total domination and exact diameter. The vertices of Γ(Zn) are partitioned by their gcd with n, and two gcd classes are completely joined if the gcd n divides de, but a class is complete internally if the gcd n divides d^2. This yields χ(Γ(Zn)) = ω(Γ(Zn)) = a - 1 + ω(b). If the number n is a prime power or n = 2p with p an odd prime, then both the domination number and the connected domination number are 1; otherwise they are r. The total domination number is max{2,r} for all vertices other than Γ(Z4) that has one. The exact diameter is immediately known from the factorization pattern as 0, 1, 2 or 3. The algorithm is factorization based and without building the full graph, it is able to calculate all reported invariants; exhaustive checks for finite ranges provide support for the implementation.

Keywords: zero-divisor graph; ring of integers modulo n; clique number; chromatic number; domination; graph diameter; prime factorization

1. Introduction

It is a useful way to encode the relations of multiplication, annihilation and ideals into combinatorial data, by associating a graph with an algebraic structure. The concept of zero-divisor graphs was introduced by Beck, in connection with a coloring problem on a commutative ring [1]. Anderson and Naseer further developed the coloring viewpoint [2] and Anderson and Livingston developed the currently widely used simple graph whose vertices are the nonzero zero divisors, with a zero product of the two vertices of an edge being recorded as an edge [3]. This graph is called Γ(R) for a commutative ring R with identity. It is joined when it is not empty and its diameter is at most 3 [3]. These basic global data led to a lot of study on girth, planarity, metric dimension, spectra, domination and other concepts with ideals or annihilator classes as starting points [4-12].

The ring Zn is the most convenient family of test rings, since the zero divisors, ideals, and annihilators of the ring Zn are determined by divisors of n. However, it can be huge: in its construction, it is necessary to make an n - φ(n) - 1 connections, which is inefficient even if the prime factorization is already known for n. A more informative representation puts residues x into classes d = gcd(x,n). Divisor classes were used by Young to partition adjacency matrices [13] and similarly by related compressed constructions, equal-annihilator elements are replaced by just one vertex [8,9,14]. In recent years, efforts have been made to characterize compressed graphs by distance-based parameters like outer multi set dimension [15]. If you look at the attached reference paper by Riaz et al. [15] for instance, you can see how annihilator compression allows to see the algebraic structure while reducing repeated neighbourhoods. The present study considers the classical graph, however, since the clique and domination parameters are related to multiplicities within gcd classes, the classical graph is kept.

There are a few exact results for Γ(Zn) but scattered. Its diameter, girth and chromatic number were characterized by Pi, Kim and Lim [16]. AbdAlJawad and Al-Ezeh found the ordinary domination number [17]. Tian and Li reconsidered the notion of clique number and provided an updated constructive procedure [18]. Several metric or spectral quantities have been considered separately as well as prime-power structure [19-22]. It would therefore be incorrect to say that these invariants have not been known individually. The true method difference is the lack of a short, standard derivation that begins with the prime factorisation, maintains class multiplicity, and then establishes the principal formulas in one theorem string and outputs an implementation that doesn't materialise every n residue.

This paper is designed to fill this gap. It's four-fold contribution. First, it provides a gcd-class decomposition which gives order, degree and size formulas and is a standard proof tool. Secondly, it explains the equality of clique and chromatic number as a square part $n = a^2b$. Thirdly, it gives ordinary, connected and total domination the same factorisation table and short proofs of the second and third consequences. Fourth, it calculates the exact diameter based on a common neighbour measure, and provides pseudocode that is cost-effective primarily based on the number of factors and number of divisors, not on the order of the graph. This report is designed to serve as a benchmark for the further development of invariant calculation in residue-class rings, product rings, compressed rings and ideal-based rings.

Table 1. Position of the study relative to selected literature.

Source	Graph or parameter	Role in the present study
Beck [1]	Ring coloring graph	Historical origin of the coloring problem.
Anderson-Livingston [3]	Classical Γ(R); connectivity	Standard vertex and adjacency convention; diameter bound.
Young [13]	Divisor-class adjacency matrix	Supports the gcd-class partition and weighted quotient view.
Pi et al. [16]	Diameter, girth, chromatic number	Provides exact cases integrated into one factorisation framework.
AbdAlJawad-Al-Ezeh [17]	Domination and independence	Source for the ordinary domination classification.
Tian-Li [18]	Clique construction	Corrected constructive basis for the clique formula.
Riaz et al. [15]	Compressed graph; OMSD	Recent comparison showing why compression differs from the classical graph.

2. Preliminaries and problem formulation

All rings are finite, commutative and with identity. Graphs are finite, simple and undirected. Suppose $n \geq 4$ is a composite number and let Zn = Z/nZ be the cyclic group. The classical zero-divisor graph of Zn is the graph Γ(Zn) with vertex set Z*(Zn) = {x in Zn : x is not 0 and gcd(x,n) > 1} in which two vertices x and y are connected if x y is congruent to 0 (modulo n). Let |V|, |E|, ω, χ, γ, γc, γt, and diam represent the order, size, clique number, chromatic number, domination number, connected domination number, total domination number, and diameter of the graph, respectively. If there is no vertex with a neighbour, then γt is undefined, as in the case of the one-vertex graph Γ(Z4).

Write the prime factorisation of n as n = product from i = 1 to r of pi to αi where $\alpha i \geq 1$ and the pi are all different. Let a = πi raised to the power of floor(αi/2), and b = πi raised to the power of 1 over the indices for which αi is odd.Define a = product of πi raised to the power of floor(αi/2) and b = product of πi raised to the power of 1 over the indices where αi is odd. If b is square-free then gcd information can be cleanly split into even and odd valuation layers and $n = a^2b$. Again, let h = ω(b) where here ω(m) refers to the number of prime factors of the number m; as with the graph clique number, no confusion should arise if it was written as ω(Γ(Zn)).

$$n = \prod_{i=1}^{r} p_i^{\alpha_i} = a^2b, \quad a = \prod_{i=1}^{r} p_i^{\lfloor \alpha_i/2 \rfloor}, \quad b = \prod_{\{\alpha_i \text{ odd}\}} p_i, \quad h = \omega(b). \quad (1)$$

Let n be a positive integer, and for every positive integer d with 1< d < n, let Cd = {x in Zn : gcd(x, n) = d} be the gcd class of x in Zn.Let n be a positive integer, and for each positive integer d with 1< d < n, let Cd = {x in Zn : gcd(x, n) = d} be the gcd class of x in Zn. The vertex set is partitioned in these classes. The cardinalities are based on Euler's totient function as x = du and u is a unit modulo n/d.

$$|C_d| = \varphi(n/d). \quad (2)$$

Now the research question can be clearly formulated: What can be calculated from the factorisation data (pi, αi) and the induced divisor classes without having to count all the residues? What is important is that all classes are adjacent in a uniform manner. This property is implemented in the same finite divisor lattice, applies to exact counting, extremal arguments and distance analysis.

3. Methodology and analytical framework

3.1 Gcd-class reduction

The variable x should be in Cd and y in Ce. Since x/d and y/e are units relative to the complementary prime powers which are still present in n/d and n/e, respectively, they can not provide a missing p-adic valuation. Hence n divides xy if and only if n divides de. It is easy to see that there is no edge between any two different classes or they are completely merged together. Two elements a and b in Cd are adjacent if and only if n divides d^2. Γ(Zn) is a weighted blow-up of a graph with its vertices being the proper divisors of the number n. The weight of each divisor node d is ϕ(n/d); if n divides d2, then d is replaced by a clique; otherwise, it is replaced by an independent set.

$$C_d \text{ is completely joined to } C_e \Leftrightarrow n \mid de; \quad \Gamma(Z_n)[C_d] \text{ is complete} \Leftrightarrow n \mid d^2. \quad (3)$$

This is the main reduction for this lemma. It keeps all the multiplicities required for clique, coloring and domination, and the compressed annihilator graph keeps only one representative from each class. This distinction is why a parameter computed from the compressed graph can't in general be simply moved to Γ(Zn).

3.2 Order, degree, and size

The units of Zn are exactly the φ(n) residue classes coprime to n. Removing the units and zero from all n residues gives the order formula. For x in Cd, the annihilator of x in Zn contains exactly d elements. After removing zero, it contributes d - 1 possible neighbours; if $x^2 = 0$, then x itself must also be removed because the graph is simple. Therefore the degree is constant on each gcd class.

$$|V(\Gamma(Z_n))| = n - \varphi(n) - 1; \quad \deg(x) = d - 1 - 1_{\{n|d^2\}} \text{ for } x \in C_d. \quad (4)$$

The class-to-class edges can be summed to get an exact size expression. For a positive integer n, define the set D(n) to be the set of all positive integers d, such that 1 < d < n. The sum below the first is the number of complete joins of different classes and the sum below the second is the number of internal edges in square zero classes.

$$|E| = \Sigma_{\{d<e,\ d,e \in D(n),\ n|de\}} \varphi(n/d)\varphi(n/e) + \Sigma_{\{d \in D(n),\ n|d^2\}} C(\varphi(n/d),2). \quad (5)$$

Equations (2)-(5) avoid an $O(|V|^2)$ pairwise product test. If the divisors of n are known, at most (τ(n)-2 choose 2) divisibility tests can be performed to determine the size, using the divisor counting function τ.

3.3 Factorisation-driven computation

There are two levels to the computational method. The closed-form level computes a, b, h, r directly from the primary exponents and provides the value of ω, χ, domination parameters and diameter. The divisor-class level counts only proper divisors to generate class sizes, degree and a precise number of edges. The following pseudocode is designed to be language-neutral.

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Algorithm 1  Major invariants of Γ(Zn)
Input: prime factorisation n = ∏ pi^αi, n composite
1. r ← number of distinct pi; a ← ∏ pi^floor(αi/2); h ← number of odd αi
2. order ← n - φ(n) - 1; clique ← chromatic ← a - 1 + h
3. if r = 1 or n = 2p (p odd prime), domination ← connected domination ← 1; else ←
r
4. if n = 4, total domination ← undefined; else total domination ← max(2,r)
5. if n = 4, diameter ← 0; else if n = p² with p ≥ 3, diameter ← 1
6. else if n = p^k (k ≥ 3) or n = pq (p ≠ q), diameter ← 2; else diameter ← 3
7. enumerate proper divisors d; set wd ← φ(n/d); apply Eq. (5) for size
Output: order, size, clique, chromatic, domination parameters, diameter
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4. Main results

4.1 Clique and chromatic numbers

The factor a is the greatest square that divides n, and h marks the prime powers that do not. The max-clique and the minimum proper colouring are completely determined by these two quantities.

Theorem 1. Suppose that $n = a^2b$ has $h = \omega(b)$ and that b is square-free. Let G be the zero-divisor graph of Zn.Let G be the zero-divisor graph of Zn.

$$\omega(\Gamma(Zn)) = \chi(\Gamma(Zn)) = a - 1 + h. \quad (6)$$

Proof. Consider first the a - 1 nonzero multiples of ab: ab, 2ab, ..., (a - 1)ab. Any two of the collection form a clique because their product is divisible by $a^2b = n$. If b is divisible by a prime q, then add bq to the list. For each such residue $r = a^2 \pmod n$ and $r' = b^2 \pmod n$ there are two such residues, one with residue $a^2b^2 \pmod n$ and one with residue $a^2b^2 \pmod{n'}$ and each is adjacent to each nonzero multiple of ab. Therefore, there is a clique of size a - 1 + h.

Encode a residue by its vector of "truncated" p-adic valuations for the upper bound. A maximum clique can be normalized to the highest valuation of its clique by swapping all the elements with a valuation greater than half the highest valuation with the complete set of non-zero multiples of ab. If a vertex is not a part of the clique, then it should be below the ceiling in an odd exponent coordinate. A pair of such zero coordinates would make these two coordinates different, as otherwise two vertices would provide the same prime pi, αi times. The h vertices are not Coordinate Zero, except at those h vertices, so all other vertices (h - 1) are not Coordinate Zero. This is the valuation form of the corrected constructive clique argument in [18]. Hence $\omega \le a - 1 + h$.

Pi et al. used the partition of the valuation types at the half-exponent thresholds [16] to obtain a coloring which is the same number of colors. Alternatively, color the square-zero core with a - 1 different colors and label each other divisor class by its first deficient odd coordinate and the normalised residue for the square part. The divisibility criterion in Eq. (3) ensures that no two adjacent vertices are the same color. Therefore $\chi \le a - 1 + h$. For any graph, one can show that $\omega = \chi$ since $\omega \le \chi$. Corollary 1.

When n is square free with r prime factors, a = 1 and h = r and hence $\chi = \omega = r$. If all exponents are even, then b = 1 and $\chi = \omega = \sqrt{n} - 1$. For a prime power n = p^k, the common value is p^{k/2} - 1 when k is even and p^{(k-1)/2} when k is odd.

4.2 Ordinary, connected, and total domination

A set S is dominating a graph if every vertex in the graph not in S has a neighbour in S. It is connected dominating if the induced graph on S is connected, it is total dominating if every vertex of S, including the vertices of S, has a neighbour in S. Suppose that a positive number n has r prime factors.

Theorem 2. The ordinary domination number for composite n is

$$\gamma(\Gamma(Zn)) = 1 \text{ if } n \text{ is a prime power or } n = 2p \text{ with } p \text{ odd prime; otherwise } \gamma(\Gamma(Zn)) = r. \quad (7)$$

The cases of prime power and n = 2p follow from a universal neighbour, namely p^{k-1} is a power of p that dominates $\Gamma(Z_{p^k})$ and p is a power of p that dominates $\Gamma(Z_{2p})$. Otherwise the other side S = {n/p1,..., n/pr} dominates. In fact, for every x in the set of zero divisors, there exists at least one pi such that x(n/pi) is equal to zero mod n for n/pi, but not necessarily for x. Consider suitable prime-supported test vertices for the lower bound. The vertex on one side of the graph cannot annihilate test vertices on two different side of the graph, except for the case of 2p with an exceptional bipartite graph. So a minimum of r options is required. The exact classification of [17] is so obtained.

Corollary 2. For every composite n the connected domination number is equal to the ordinary domination number. By convention a singleton dominating set is connected. In the other cases, the set S above is a clique since (n/pi)(n/pj) is an integer for all $i \neq j$. This means that the minimum dominating set already has a connected realization.

Corollary 3. If n is not 4, then γt(Γ(Zn)) = max {2,r}. The canonical set S is dominating and pairwise adjacent for r>2 and hence it is a total dominating set. For $r \geq 3$, $|S| \geq r$, and for r = 2, any total dominating set contains an ordinary dominating set of size 2. If the power p^k is a prime power, use p and p^{k-1} for $k \geq 3$; if it is an odd prime square, use the two adjacent vertices p and 2p. This means that two vertices are enough and one is never enough. No total dominating set exists in the graph Γ(Z4) which is K1.

4.3 Exact diameter

A more stringent common-neighbour test leads to the diameter classification. Let x be in Cd and y be in Ce. A residue z works on x and y only when it is a multiple of lcm(n/d,n/e). Since

$$lcm(n/d,n/e) = n/gcd(d,e), \quad (8)$$

The gcd(d,e) > 1 condition holds true if and only if a common neighbour exists between d and e (excluding x and/or y if a simple-graph loop should be used). If gcd(d,e) = 1 and n is not a divisor of de, then the vertices are nonadjacent and do not share a common neighbour, their distance in Γ(Zn) is therefore three (Γ(Zn) has diameter three [3]).

Theorem 3. The exact diameter is given by the following exhaustive alternatives.

$$\operatorname{diam} \Gamma(Zn) = 0, \text{ if } n = 4;$$

$$\operatorname{diam} \Gamma(Zn) = 1, \text{ if } n = p^2 \text{ with prime } p \geq 3;$$

$$\operatorname{diam} \Gamma(Zn) = 2, \text{ if } n = p^k \text{ with } k \geq 3, \text{ or } n = pq \text{ with distinct primes } p,q;$$

$$\operatorname{diam} \Gamma(Zn) = 3, \text{ in every other composite case.} \quad (9)$$

The graph is complete if this prime square is a prime square, and has positive diameter one for $p \geq 3$; p = 2 yields K1. If n>2, then all the gcd classes contain p, and then any two nonadjacent classes have a common annihilator, therefore the diameter is 2. If n = pq, then Γ(Zn) is the complete bipartite graph $K_{q-1,p-1}$ of diameter two as well. But in all other cases with at least two different prime numbers, select coprime proper factor d, e of n; the representatives in Cd and Ce then have a distance of three by Eq. (8). This yields the same form of case analysis as in [16] given one arithmetic criterion.

5. Worked examples and computational verification

5.1 Detailed example: n = 72

Let $n = 72 = 2^3 3^2$. Then $a = 2^1 3^1 = 6$, $b = 2$, $h = 1$, and $r = 2$. According to Euler's function the graph has 47 vertices, since $\varphi(72) = 47$. Equation (5) gives 136 edges. The clique number and chromatic number are equal to $6 - 1 + 1 = 6$. As 72 is not a prime power, nor twice an odd prime, $\gamma = \gamma c = 2$, and the two vertices 36 and 24 form a connected dominating set. The total domination number is also 2. Lastly, $d = 8$ and $e = 9$ are coprime, with $de = 72$, and are adjacent; however, $d = 2$ and $e = 3$ are coprime with $de = 6$ and have distance-three representatives, thus the diameter is 3.

Table 2. Gcd-class structure of Γ(Z72).

d	|C_d| = φ(72/d)	Internal type	Completely joined to higher-index classes
2	12	Independent	36
3	8	Independent	24
4	6	Independent	18, 36
6	4	Independent	12, 24, 36
8	6	Independent	9, 18, 36
9	4	Independent	24
12	2	Complete	18, 24, 36
18	2	Independent	24, 36
24	2	Complete	36
36	1	Complete	None

5.2 Cross-family comparison

The invariants are summarized in Table 3 with respect to the parity of the exponents and the number of different prime factors. Complete graphs are generated by the prime squares 9 and 49. For larger prime powers, the dominance number is still one but the diameter is increased to two since not all pairs are adjacent. If a prime factor occurs more than once in the product or the product is not square-free, then there are usually coprime divisor types at distance three; this is a rule that can be broken by a third prime factor, however. The clique number can increase very quickly even when the domination number is low, for example, for a graph that is a square factor.

Table 3. Exact invariants for representative composite moduli.

n	Prime factorisation	|V|	|E|	ω = χ	γ = γc	γt	diam
4	2^2	1	0	1	1	N/A	0
8	2^3	3	2	2	1	2	2
9	3^2	2	1	2	1	2	1
10	$2\cdot5$	5	4	2	1	2	2
12	$2^2\cdot3$	7	8	2	2	2	3
15	$3\cdot5$	6	8	2	2	2	2
16	2^4	7	7	3	1	2	2
18	$2\cdot3^2$	11	13	3	2	2	3
27	3^3	8	13	3	1	2	2
30	$2\cdot3\cdot5$	21	38	3	3	3	3

n	Prime factorisation	|V|	|E|	ω = χ	γ = γc	γt	diam
36	$2^2 \cdot 3^2$	23	46	5	2	2	3
49	7^2	6	15	6	1	2	1
72	$2^3 \cdot 3^2$	47	136	6	2	2	3
90	$2 \cdot 3^2 \cdot 5$	65	193	4	3	3	3
210	$2 \cdot 3 \cdot 5 \cdot 7$	161	668	4	4	4	3
360	$2^3 \cdot 3^2 \cdot 5$	263	1528	7	3	3	3

5.3 Verification protocol

Exhaustive finite enumeration was used to independently check the formulas. The reference implementation was used to compute all residues x such that $1 < \gcd(x,n)$ and to add an edge if the product $xy \bmod n = 0$, for each n. We compared order, degree sequence, edge count and diameter with Eqs. (4), (5), and (9). For small n ($4 \le n \le 60$), the maximum cliques were calculated on the full graph, while for larger n ($4000 \le n \le 2000$), they were computed on the weighted divisor-class graph. The chromatic numbers were computed exactly with branch-and-bound coloring for $4 \le n \le 60$, the domination numbers were checked with subset enumeration for $4 \le n \le 120$, and the weighted clique formula was checked for all composite n in the range $4000 \le n \le 2000$. There was no difference noted. The following calculations are only for checking purposes and are not to be used in place of the proofs. They are particularly helpful in the case of $n = 4$, in cases of loops, and in cases of convention errors.

6. Discussion

6.1 What the unified factorisation reveals

The formulas are used to differentiate three kinds of arithmetic information. The large square-zero core is controlled by the square-root divisor a, and so is the dominant part of χ and ω. The parity count h provides an extra 1 clique/coloring unit for each odd prime power layer. Domination, on the other hand, is mainly based on the number of different prime exponents, r, and is not very sensitive to the values of most of the exponents. Also coarser is diameter: given that the cases are not complete squares, does the modulus factorize into the product of two different prime numbers, or does it have a more complex mixed factorization. So a single prime factorisation generates profiles of varying resolution that are invariant.

This separation also accounts for the fact that there are two different structures of graphs with the same number of vertices: six is the number of vertices in $\Gamma(Z49)$ and six is the number of vertices in $\Gamma(Z15)$, which is complete bipartite. They have the same orders but different exponent patterns for clique numbers, domination behaviour and diameters. Similarly, the number of prime supports for $\Gamma(Z72)$ is only two, and hence the domination number of it is two, but the clique number of its half-exponent square part is six. The parameters should hence not be understood as synonyms for measuring complexity.

6.2 Classical versus compressed graphs

If gcd classes are the same in Zn, the elements of the same gcd class have the same annihilator and have almost the same open neighbourhoods, except for the case of a square-zero vertex itself treated as a simple graph. For each class, compression brings down the divisor multiplicity $\varphi(n/d)$ to 1. This is beneficial for invariants used to differentiate the types of annihilators, such as outer multiset dimension studies [15] but it alters the size of the clique when more than one element is in a square-zero class. It can also change the domination since it no longer represents several vertices that may have to be adjacent within the same class. This weighted blow up allows us to have a “bridge”: it is compact but is able to capture precisely the multiplicities and the internal clique/independent-set status essential for the reconstruction of classical invariants.

6.3 Computational implications

Direct graph construction examines up to (n - $\varphi(n)$ - 1 choose 2) pairs and puts one graph of linear size in the worst relevant cases into memory. The factorisation route calculates the main closed forms with the number of arithmetic operations: O(r) after factorisation. Only $\tau(n)$ - 2 divisor classes will be processed, if the exact edge count and degree distribution is needed. This change is important for moduli that have large prime factors, because the residue graph may be large, but the divisor lattice may not be. The approach is also repeatable: Every result can be verified using the prime-exponent vector, without having to rely on a plotted graph or manually generated adjacency list.

6.4 Limitations and future directions

The analysis is limited to the simple zero-divisor graph of Zn. Does not contain the vertex 0, does not have loops if $x^2 = 0$, and should not be transferred (unchanged) to Beck's original graph, total graphs, exact zero-divisor graph, or extended zero-divisor graph. The Chinese remainder theorem decomposes Zn into a product of prime-power rings but there are other finite commutative rings in which the multiplicities of the annihilator classes are not necessarily totient values. These present formulas must be given new local-ring input before extension.

There are a number of obvious directions to take from here. Under the same notation, the weighted divisor-class model can be applied to obtain the formulas of exact independence, matching, metric-dimension and resolving-domination. Secondly, multivariate generating functions may be used to encode the degree and distance distributions of the exponent vectors. Third, there should be developed algorithms for products of residue-class rings, and also for Gaussian integer quotients, where the types of valuations remain but the class weights change. Finally, by comparing Γ(Zn) and its compressed version using invariant-preserving graph operations, it may become clear exactly which parameters are preserved by such compression of the annihilator. Finally, it would be possible to use a formal proof assistant or computer-algebra system to verify the divisor-class decomposition and to generate automatically case tables for a given factorisation to prove the theorems.

7. Conclusion

These apparent 'fragmentation' of major invariants of Γ(Zn) can be resolved by analyzing the classes of gcd. The class Cd contains $\varphi(n/d)$ vertices, and has uniform degree exactly when n divides d^2, and is completely joined to Ce exactly when n divides de. Based on these facts, the notions of order, size, clique, coloring, domination, and distance becomes arithmetic statements on the divisor lattice. If we write $n = a^2b$, where b is square-free, then we have the compact equality $\chi = \omega = a - 1 + \omega(b)$. For prime powers, and for n = 2p, ordinary and connected domination are equal, otherwise they equal the number of distinct primes. For every z>4, the domination number of z is max(2,r). The common-neighbour identity lcm(n/d,n/e) = n/gcd(d,e) provides the exact characterization of the diameter-classification and precisely when distance three happens. What is valuable is not that every formula is new, it is actually a coherent derivation, a correct attribution, other domination consequences, and an efficient implementation based on factorisation. The arithmetic mechanisms can be made explicit in this organisation, which can serve as a starting point to explore other invariants of classical, compressed, and ideal based zero-divisor graphs.

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Author contributions:

Kannan M - Conceptualization, methodology, formal analysis and writing original draft and review.

Sasikala A – review, methodology, verification and formal analysis.

References

[1] Beck I. Coloring of commutative rings. Journal of Algebra. 1988;116(1):208-226. doi:10.1016/0021-8693(88)90202-5.

[2] Anderson DD, Naseer M. Beck's coloring of a commutative ring. Journal of Algebra. 1993;159(2):500-514.

[3] Anderson DF, Livingston PS. The zero-divisor graph of a commutative ring. Journal of Algebra. 1999;217(2):434-447. doi:10.1006/jabr.1998.7840.

[4] Lucas TG. The diameter of a zero-divisor graph. Journal of Algebra. 2006;301(1):174-193.

[5] Anderson DF, Mulay SB. On the diameter and girth of a zero-divisor graph. Journal of Pure and Applied Algebra. 2007;210(2):543-550. doi:10.1016/j.jpaa.2006.10.007.

[6] Redmond SP. On zero-divisor graphs of small finite commutative rings. Discrete Mathematics. 2007;307(9-10):1155-1166.

[7] Redmond SP. An ideal-based zero-divisor graph of a commutative ring. Communications in Algebra. 2003;31(9):4425-4443.

[8] Mulay SB. Cycles and symmetries of zero-divisors. Communications in Algebra. 2002;30(7):3533-3558.

[9] Spiroff S, Wickham C. A zero divisor graph determined by equivalence classes of zero divisors. Communications in Algebra. 2011;39(7):2338-2348.

[10] Anderson DF, LaGrange JD. Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph. Journal of Pure and Applied Algebra. 2012;216(7):1626-1636.

[11] Anderson DF, LaGrange JD. Some remarks on the compressed zero-divisor graph. Journal of Algebra. 2016;447:297-321.

[12] Axtell M, Coykendall J, Stickles J. Zero-divisor graphs of polynomials and power series over commutative rings. Communications in Algebra. 2005;33(6):2043-2050. doi:10.1081/AGB-200063357.

[13] Young M. Adjacency matrices of zero-divisor graphs of integers modulo n. Involve. 2015;8(5):753-761. doi:10.2140/involve.2015.8.753.

[14] Mönius K. Eigenvalues of zero-divisor graphs of finite commutative rings. Journal of Algebraic Combinatorics. 2021;54:787-802. doi:10.1007/s10801-020-00989-6.

[15] Riaz A, Siddiqui HMA, Riaz MB, Ali N. Graph-theoretic characterization of rings: Outer multiset dimension of compressed zero-divisor graphs. Ain Shams Engineering Journal. 2025;16:103693. doi:10.1016/j.asej.2025.103693.

[16] Pi SJ, Kim SH, Lim JW. The zero-divisor graph of the ring of integers modulo n. Kyungpook Mathematical Journal. 2019;59(4):591-601. doi:10.5666/KMJ.2019.59.4.591.

[17] AbdAlJawad EE, Al-Ezeh H. Domination and independence numbers of Γ(Zn). International Mathematical Forum. 2008;3(11):503-511.

[18] Tian Y, Li L. Comments on the clique number of zero-divisor graphs of Zn. Journal of Mathematics. 2022;2022:6591317. doi:10.1155/2022/6591317.

[19] Pirzada S, Aijaz M, Bhat MI. On zero divisor graphs of the rings Zn. Afrika Matematika. 2020;31:727-737. doi:10.1007/s13370-019-00755-3.

[20] Chattopadhyay S, Patra KL, Sahoo BK. Laplacian eigenvalues of the zero divisor graph of the ring Zn. Linear Algebra and its Applications. 2019;584:267-286.

[21] Pirzada S, Raja R. On the metric dimension of a zero-divisor graph. Communications in Algebra. 2017;45(4):1399-1408.

[22] Pirzada S, Bhat MI. Computing metric dimension of compressed zero divisor graphs associated to rings. Acta Universitatis Sapientiae, Mathematica. 2018;10(2):298-318.

[23] Aalipour G, Akbari S, Nikandish R, Nikmehr MJ, Shaveisi F. On the coloring of the annihilating-ideal graph of a commutative ring. Discrete Mathematics. 2012;312(17):2620-2626.

[24] Elahi K, Ahmad A, Hasni R. Construction algorithm for zero divisor graphs of finite commutative rings and their vertex-based eccentric topological indices. Mathematics. 2018;6(12):301. doi:10.3390/math6120301.