On The Quinary Homogeneous Cubic Polynomial Diophantine Equation

$$x^3 + y^3 + (x+y)(x-y)^2 = 23(z+w)\,P^2$$

N.Thiruniraiselvi[1], S.Devibala[2], M.A.Gopalan[3]

[1]Associate Professor, Department of Mathematics,
Sri Vignesh College of Engineering and Technology,
(Affiliated to Anna University, Chennai), Koothur, Trichy- 621 216,Tamil Nadu, India.
Email: drntsmaths@gmail.com
(https://orcid.org/0000-0003-4652-3846)

[2]Associate Professor, Department of Mathematics, Sri Meenakshi Government Arts College for Women(A), Affiliated to Madurai Kamaraj University, Madurai-625 002, Tamil Nadu, India.
devibalasubburaman@gmail.com
https://orcid.org/0000-0003-1110-0587

[3]Professor of Mathematics, Shrimati Indira Gandhi College,
Affiliated to Bharathidasan University, Trichy-620 002,
Tamil Nadu, India.
mayilgopalan@gmail.com
https://orcid.org/0000-0003-1307-2348

Abstract

The homogeneous cubic Diophantine equation with five unknowns represented by $x^3 + y^3 + (x+y)(x-y)^2 = 23(z+w)P^2$ is analyzed for its patterns of non – zero distinct integral solutions.

Keywords: Cubic Equation, Homogeneous cubic equation, Quinary cubic equation, Integral Solutions.

Introduction

The subject of Diophantine equations is a treasure house in which the search for many hidden connections is a treasure hunt. In other words, the theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers. It is worth to observe that cubic Diophantine equations fall into the theory of Elliptic curves which are used in Cryptography. Many third degree Diophantine equations in two or three variables can be represented as elliptic curves that are used to generate public and private key pairs. Cubic Diophantine equations such as $x^3 + y^3 = z^3 + w^3$ requiring integer solutions form the backbone of many advanced mathematical and computational fields. Researchers utilize cubic equation with four variables such as $x^3 + y^3 = 2nzw^2$ to produce sequences of figurate numbers. For simplicity and brevity, refer [1-23] for various problems on the cubic diophantine equations .However, often we come across homogeneous and non-homogeneous cubic equations and as such one may require its integral solution in its most general form. Here, the problem of obtaining solutions in integers to the third degree polynomial equation having five unknowns given by $x^3 + y^3 + (x+y)(x-y)^2 = 23(z+w)P^2$ is studied.

Method of analysis

The homogeneous cubic polynomial diophantine equation with five unknowns to be solved for its non-zero distinct integral solutions is given by

$$x^3 + y^3 + (x+y)(x-y)^2 = 23(z+w)P^2 \quad (1)$$

Introducing the linear transformations

$$x = u+v, y = u-v, z = u+1, w = u-1, u \neq \pm 1, v \quad (2)$$

in (1), leads to

$$u^2 + 7v^2 = 23P^2 \quad (3)$$

To start with, by inspection, observe that (3) is satisfied by

$$u = 4*23\alpha, v = 23\alpha, P = 23\alpha$$

In view of (2), the values of x ,y, z and w satisfying (1) are given by

$$x = 5*23\alpha, y = 3*23\alpha, z = 4*23\alpha + 1, w = 4*23\alpha - 1$$

Now, we present below different methods of solving (3) and thus obtain different Patterns of integral solutions to (1).

Pattern 1

Write (3) as

$$23P^2 - u^2 = 7v^2 \quad (4)$$

Assume

$$v = 23a^2 - b^2 \quad (5)$$

Consider the integer 7 on the R.H.S. of (4) to be

$$7 = (\sqrt{23} + 4)\,(\sqrt{23} - 4) \quad (6)$$

Substitute (5) & (6) in (4). Employing factorization and equating the positive factors, one has

$$\sqrt{23}\,P + u = (\sqrt{23} + 4)(\sqrt{23}\,a + b)^2$$

On equating the rational and irrational parts, one obtains

$$\left.\begin{aligned} u &= 4(23a^2 + b^2) + 46ab \\ P &= (23a^2 + b^2) + 8ab \end{aligned}\right\} \quad (7)$$

Using the values of u ,v from (7) ,(5) in (2), one gets

$$\begin{aligned} x &= 5*23a^2 + 3b^2 + 46ab \\ y &= 3*23a^2 + 5b^2 + 46ab \\ z &= 4(23a^2 + b^2) + 46ab + 1 \\ w &= 4(23a^2 + b^2) + 46ab - 1 \end{aligned} \tag{8}$$

Thus, the values of x, y, z, w and P given by (8) and (7) satisfy (1).

Pattern 2

Write (3) as

$$23P^2 - 7v^2 = u^2 * 1 \tag{9}$$

Assume

$$u = 4(23a^2 - 7b^2) \tag{10}$$

Consider the integer 1 on the R.H.S. of (9) to be

$$1 = \frac{(\sqrt{23}+\sqrt{7})(\sqrt{23}-\sqrt{7})}{16} \tag{11}$$

Substitute (10) & (11) in (9). Employing factorization and equating the positive factors, one has

$$\begin{aligned} \sqrt{23}P + \sqrt{7}v &= (2\sqrt{23}a + 2\sqrt{7}b)^2 \frac{(\sqrt{23}+\sqrt{7})}{4} \\ &= (\sqrt{23}a + \sqrt{7}b)^2 (\sqrt{23}+\sqrt{7}) \end{aligned}$$

On equating the coefficients of corresponding terms in the above equation, one obtains

$$\begin{aligned} v &= (23a^2 + 7b^2) + 46ab \\ P &= (23a^2 + 7b^2) + 14ab \end{aligned} \tag{12}$$

Using the values of u ,v from (10) ,(12) in (2), one gets

$$\begin{aligned} x &= 5*23a^2 - 3*7b^2 + 46ab \\ y &= 3*23a^2 - 5*7b^2 - 46ab \\ z &= 4(23a^2 - 7b^2) + 1 \\ w &= 4(23a^2 - 7b^2) - 1 \end{aligned} \tag{13}$$

Thus, the values of x, y, z, w and P given by (13) and (12) satisfy (1).

Pattern 3

Write (3) as

$$u^2 = 23P^2 - 7v^2 \tag{14}$$

Insertion of the transformations

$$P = X + 7T,\ v = X + 23T,\ u = 4U \tag{15}$$

in (14) leads to

$$X^2 = U^2 + 161T^2 \tag{16}$$

which is satisfied by

$$T = 2rs,\ U = 161r^2 - s^2,\ X = 161r^2 + s^2$$

From (15), one has

$$\begin{aligned} &u = 644r^2 - 4s^2,\ v = 161r^2 + s^2 + 46rs \\ &P = 161r^2 + s^2 + 14rs \end{aligned} \tag{17}$$

From (2), we have

$$\begin{aligned} x &= 805r^2 - 3s^2 + 46rs \\ y &= 483r^2 - 5s^2 - 46rs \\ z &= 644r^2 - 4s^2 + 1 \\ w &= 644r^2 - 4s^2 - 1 \end{aligned} \tag{18}$$

Thus, the values of x, y, z, w and P given by (18) and (17) satisfy (1).

Note 1

It is worth to mention that, apart from (15),one may consider the transformation as

$$P = X - 7T,\ v = X - 23T,\ u = 4U$$

For this choice, we get

$$\begin{aligned} &u = 644r^2 - 4s^2,\ v = 161r^2 + s^2 - 46rs \\ &P = 161r^2 + s^2 - 14rs \end{aligned} \tag{19}$$

The corresponding integer solutions to (1) are

$$\begin{aligned} x &= 805r^2 - 3s^2 - 46rs \\ y &= 483r^2 - 5s^2 + 46rs \\ z &= 644r^2 - 4s^2 + 1 \\ w &= 644r^2 - 4s^2 - 1 \end{aligned} \tag{20}$$

together with P given by (19).

Further, (16) is written as the system of double equations as below in Table-1:

Table-1 System of double equations

System	I	II	III	IV	V	VI
$X+U$	T^2	$7T^2$	$23T^2$	$161T^2$	$23T$	$161T$
$X-U$	161	23	7	1	$7T$	T

Solving each of the above system of double equations, one obtains the values of X,U and T. Substituting these values in (15) and (2), the required integer solutions to (1) are obtained. For simplicity & brevity, we present below the respective six sets of integer solutions to (1):

Set 1: $x = 10s^2 + 52s - 216, y = 6s^2 - 44s - 424, z = 8s^2 + 4s - 319$
$w = 8s^2 + 4s - 321, P = 2s^2 + 16s + 88$

Set 2: $x = 70s^2 + 116s + 6, y = 42s^2 - 4s - 70, z = 56s^2 + 56s - 31$
$w = 56s^2 + 56s - 33, P = 14s^2 + 28s + 22$

Set 3: $x = 230s^2 + 276s + 70, y = 138s^2 + 92s - 6, z = 184s^2 + 184s + 33$
$w = 184s^2 + 184s + 31, P = 46s^2 + 60s + 22$

Set 4: $x = 1610s^2 + 1656s + 423, y = 966s^2 + 920s + 217, z = 1288s^2 + 1288s + 321$
$w = 1288s^2 + 1288s + 319, P = 322s^2 + 336s + 88$

Set 5: $x = 102T, y = 26T, z = 64T + 1, w = 64T - 1, P = 22T$

Set 6: $x = 424T, y = 216T, z = 320T + 1, w = 320T - 1, P = 88T$

Pattern 4

Write (3) in the form of ratio as

$$\frac{u + 4P}{P + v} = \frac{7(P - v)}{u - 4P} = \frac{\alpha}{\beta}, \beta > 0 \tag{21}$$

After some algebra, it is seen that (21) is satisfied by

$$\begin{aligned} u &= 4\alpha^2 - 28\beta^2 + 14\alpha\beta \\ v &= -\alpha^2 + 7\beta^2 + 8\alpha\beta \\ P &= \alpha^2 + 7\beta^2 \end{aligned} \tag{22}$$

From (2) , observe that

$$\begin{aligned} x &= 3\alpha^2 - 21\beta^2 + 22\alpha\beta \\ y &= 5\alpha^2 - 35\beta^2 + 6\alpha\beta \\ z &= 4\alpha^2 - 28\beta^2 + 14\alpha\beta + 1 \\ w &= 4\alpha^2 - 28\beta^2 + 14\alpha\beta - 1 \end{aligned} \tag{23}$$

Thus, the values of x, y, z, w and P given by (23) and (22) satisfy (1).

Note 2

It is worth to mention that (3) may also be expressed in the following ratio forms:

$$\frac{u+4P}{P-v}=\frac{7(P+v)}{u-4P}=\frac{\alpha}{\beta},\beta>0$$
$$\frac{u+4P}{7(P-v)}=\frac{(P+v)}{u-4P}=\frac{\alpha}{\beta},\beta>0$$
$$\frac{u+4P}{7(P+v)}=\frac{(P-v)}{u-4P}=\frac{\alpha}{\beta},\beta>0$$

Following the above procedure, one obtains three more patterns of integer solutions to (1).

Pattern 5

Write (3) as

$$u^2+7v^2=23P^2*1 \tag{24}$$

Assume

$$P=4(a^2+7b^2) \tag{25}$$

Consider the integers 1, 23 on the R.H.S. of (24) as

$$\begin{aligned} 1&=\frac{(3+i\sqrt{7})(3-i\sqrt{7})}{16} \\ 23&=(4+i\sqrt{7})(4-i\sqrt{7}) \end{aligned} \tag{26}$$

Substitute (25) & (26) in (24). Employing factorization and equating the positive factors, one obtains

$$\begin{aligned} u+i\sqrt{7}\,v&=(4+i\sqrt{7})(2a+i2\sqrt{7}\,b)^2\frac{(3+i\sqrt{7})}{4} \\ &=(5+i7\sqrt{7})(a+i\sqrt{7}\,b)^2 \\ &=(5+i7\sqrt{7})[f(a,b)+i\sqrt{7}\,g(a,b)] \end{aligned} \tag{27}$$

where

$$f(a,b)=a^2-7b^2\ ,g(a,b)=2ab$$

Equating the real and imaginary parts in (27), one has

$$u=5f(a,b)-49g(a,b)$$
$$v=7f(a,b)+5g(a,b)$$

From (2), we have

$$\begin{aligned} x&=12f(a,b)-44g(a,b) \\ y&=-2f(a,b)-54g(a,b) \\ z&=5f(a,b)-49g(a,b)+1 \\ w&=5f(a,b)-49g(a,b)-1 \end{aligned} \tag{28}$$

Thus, the values of x, y, z, w and P given by (28) and (25) satisfy (1).

Note 3

Apart from (26), The integer1 on the R.H.S. of (24) may be considered as below:

$$1=\frac{[7r^2-s^2+i\sqrt{7}(2rs)][7r^2-s^2-i\sqrt{7}(2rs)]}{(7r^2+s^2)^2}$$

$$1=\frac{[r^2-7s^2+i\sqrt{7}(2rs)][r^2-7s^2-i\sqrt{7}(2rs)]}{(r^2+7s^2)^2}$$

$$1=\frac{[14s^2-14s+3+i\sqrt{7}(2s-1)]\,[14s^2-14s+3-i\sqrt{7}(2s-1)]}{(14s^2-14s+4)^2}$$

$$1=\frac{[7s^2-1+i\sqrt{7}(2s)]\,[7s^2-1-i\sqrt{7}(2s)]}{(7s^2+1)^2}$$

$$1=\frac{[2s^2-2s-3+i\sqrt{7}(2s-1)]\,[2s^2-2s-3-i\sqrt{7}(2s-1)]}{(2s^2-2s+4)^2}$$

Following the above procedure, one obtains some more patterns of integer solutions to (1).

Conclusion

In this paper, an attempt has been made to obtain infinitely many integral solutions of a Cubic Diophantine equation in five unknowns. To conclude one may search for pattern of solutions to other forms of cubic equations with multiple variables.

References

1. M.A.Gopalan, S.Vidhyalakshmi,N.Thiruniraiselvi.On Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=21zw^2$,Review of Information Engineering and Applications, 01(02), p.93-101,2014.
2. M.A.Gopalan,S.Vidhyalakshmi,N.Thiruniraiselvi. On Homogeneous Cubic Equation with Four Unknowns $(x+y+z)^3=z(xy+31w^2)$, Cayley J.Math,02,p.163- 168,2013.
3. M.A.Gopalan,N.Thiruniraiselvi,V.Krithika. On the Ternary Cubic Diophantine Equation $7x^2-4y^2=3z^3$,International Journal of Recent Scientific Research,06(09),6197- 6199,2015.
4. M.A.Gopalan ,S.Vidhyalakshmi ,N.Thiruniraiselvi ,On Homogeneous Cubic Equation with Three Unknowns $x^2-y^2+z^2=2kxyz$,BOMSR,1(1),13-15,2013
5. M.A.Gopalan ,N.Thiruniraiselvi,R.Sridevi ,On the Ternary Cubic Equation $5(x^2+y^2)-8xy=74(k^2+s^2)z^3$, International Journal of Multidisciplinary Research and Modern Engineering,1(1), 317- 319,2015
6. K.Meena, M.A.Gopalan,S.Vidhyalakshmi,N.Thiruniraiselvi, On the Non - homogeneous Ternary Cubic Equation $2a^2\left(x^2+y^2\right)-2a(k+1)(x+y)+(k+1)^2=2^{2n}z^3$, Universe of Emerging Technologies and Science, II(1),1-5,2015
7. N. Thiruniraiselvi, M.A. Gopalan, A Search on Integer solutions to non-homogeneous Ternary Equation $9(x^2-y^2)+x+y=4z^3$, Epra International Journal Of Multidisciplinary Research (Ijmr) - , Vol.7 (9),15- 18,2021.

8. N. Thiruniraiselvi, M.A. Gopalan, Binary Third degree Diophantine Equation $5(x-y)^3 = 8xy$,Oriental Journal of Physical Sciences, Vol.09(1) , 73-76, 2024.
9. N. Thiruniraiselvi, Sharadha Kumar, M.A. Gopalan, Non-homogeneous Binary Cubic Equation $a(x-y)^3 = 8bxy$,Pure and Applied Mathematics Journal, Vol.8(11), 166-191, 2024
10. M.A. Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi, "On homogeneous cubic equation with four unknowns $x^3 + y^3 = 21zw^2$", Review of Information Engineering and Applications, 1(4), 93-101. 2014.
11. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, J. Maheswari, "On ternary cubic diophantine equation $3(x^2+y^2)-5xy+x+y+1=12z^3$", International Journal of Applied Research, 1(8), 209-212. 2015.
12. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, "On the cubic equation with four unknowns $x^3+4z^3=y^3+4w^3+6(x-y)^3$", International Journal of Mathematics Trends and Technology, 20(1) , 75-84. April 2015.
13. S. Vidhyalakshmi, T. Mahalakshmi, M.A. Gopalan, "A Search on the Integer Solutions of Cubic Diophantine Equation with Four Unknowns $x^3-y^3=4(w^3-z^3)+3(x-y)^3$", International Journal of Engineering And Science, 10(8), 13-17. August 2020.
14. N. Thiruniraiselvi, M.A. Gopalan, A New Class of Integer Solutions to Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=42zw^2$, International Journal of Applied Sciences and Mathematical Theory (IJASMT),10(2), 1-7, 2024.
15. N. Thiruniraiselvi, M.A. Gopalan, A Search on Integer solutions to non-homogeneous Ternary Equation $9(x^2-y^2)+x+y=4z^3$, Epra International Journal Of Multidisciplinary Research (Ijmr) - , Vol.7,Issue 9, September 2021, Pg-15- 18.
16. M.A. Gopalan, N. Thiruniraiselvi, A New Class of Integer Solutions to Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=42zw^2$, International Journal of Applied Sciences and Mathematical Theory (IJASMT), Vol. 10(2), Pg 1-7, 2024.
17. M.A. Gopalan, N. Thiruniraiselvi, Non-homogeneous Binary Cubic Equation $a(x-y)^3=8xy, a>0$, Bulletin of Pure and Applied Sciences Section - E - Mathematics & Statistics, 43E(1), 37–42, 2024.
18. M.A. Gopalan, N. Thiruniraiselvi, Techniques to solve Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=7(z-w)^2(z+w)$, International Journal of Research – GRANTHAALAYAH, 12(10), 62–69, October 2024.
19. N.Thiruniraiselvi ,Sharadhakumar,M.A.Gopalan,Non-homogeneous Quinary Cubic Equation $x^3-y^3=z^3-w^3+72t^2$,Annals of Communications in Mathematics,7(2), 95- 99,2024
20. R.Sathiyapriya, N. Thiruniraiselvi, M.A. Gopalan, A Modish Glance of Integer Solutions to Nonhomogeneous Cubic Diophantine Equation with Three Unknowns, Nanotechnology Perceptions, 21(1),(2025), 275-280, 2025.

21. N.Thiruniraiselvi,E.Premalatha,M.A.Gopalan,Non-homogeneous Ternary Cubic Diophantine Equation $5x^2-3y^2=z^3$,Bulletin of Pure and Applied Sciences-Mathematics,44(1),41-46,2025
22. N.Thiruniraiselvi,S.Devibala,M.A.Gopalan,A Sparkle of Integer Solutions to Non-homogeneous Ternary Cubic Diophantine Equation $a(x^2+y^2)=bz^3$, g.c.d.$(a,b)=1$, Foundary,29(3),212-218,2026
23. N.Thiruniraiselvi, M.A.Gopalan,A Scrutiny on the Quaternary Cubic Equation $7(x^2-y^2)=(z^3-w^3)$ 7 $(x^2-y^2)=(z^3-\ w^3)$,IRJEdT,9(5), 611-619 ,2026.