On the Super Edge Anti Magic Total Labeling for the Union of Some Graphs

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Abstract: A (p, q)-graph G is called (a, d)-edge anti magic total, (a, d)-EAMT, if there exist integers $a > 0, d \geq 0$ and a bijection $\lambda : V \cup E \rightarrow \{1, 2, \ldots, p + q\}$ such that $W = \{w(xy) : xy \in E\} = \{a, a + d, \ldots, a + (q - 1)d\}$, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ is the edge-weight of xy. An (a, d)-EAMT labeling λ of G is super, (a, d)-SEAMT, if $\lambda(V) = \{1, 2, \cdots, p\}$. An (a, d)-EAMT (SEAMT) labeling is called edge magic total (EMT) or super edge magic total (SEMT) labeling with magic constant a, if $d = 0$. A graph which have EAMT, SEAMT, EMT, or SEMT is called EAMT, SEAMT, EMT, or SEMT graph, respectively. In this paper, we show that the graphs $2P_{2n} \cup S_{n+1,n-1}$ and $C_n \cup 2K_{1,\frac{n}{2}-2}$ are EAMT, SEAMT, EMT, or SEMT which produces a new theorem that aims to there are still many unsolved graph labelling problems that make study of SEAMT graph more developed.

Keywords: Labeling, (a, d)-EAMT, (a, d)-SEAMT, Dual Labeling.

1. Introduction

We consider finite, simple, and undirected graphs, that is, graphs without loops and multiple edges. The notation V and E stand for the vertex set and edge set of graph G, respectively. Let $e = \{u, v\}$ (in short, $e = uv$) denote an edge connecting vertices u and v in G. P_n denotes a path on n vertices. We denote by (p, q) −graph G a graph with p vertices and q edges. For all standard graph-theoretic concepts and notations not explicitly defined here, we follow the conventions given in (Hartsfield and Ringel, 1994).

The labeling problems in graph theory have attracted substantial attention over the past few decades, not only due to their intrinsic mathematical elegance but also because of their numerous applications in areas such as communication networks, coding theory, and combinatorial optimization. Among these problems, antimagic labelings represent one of the most actively studied families. Roughly speaking, a graph labeling is called antimagic if it assigns distinct weights to the edges under some well-defined weight function. Within this broader framework, the notion of edge antimagic total labeling has emerged as a particularly rich and versatile area of investigation.

A (p, q)-graph G is called (a, d)-edge anti magic total, (a, d)-EAMT, if there exist integers $a > 0, d \geq 0$ and a bijection $\lambda : V \cup E \rightarrow \{1, 2, \ldots, p + q\}$ such that the set of edge-weights is $W = \{w(xy) : xy \in E\} = \{a, a + b, \ldots, a + (q - 1)d\}$, where $w(xy) = \lambda(x) + \lambda(y) +$

$\lambda(xy)$. We shall follow Ngurah and Baskoro (2003) to call $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ the edge-weight of xy, and W the set of edge-weights of the graph G. In particular, an (a, d)-EAMT labeling λ of a (p, q)-graph G is super if $\lambda(V) = \{1, 2, \cdots, p\}$. For brevity, throughout this paper we denote such a labeling by (a, d)-EAMT of G.

A number of classification studies on (a, d) −SEAMT (resp. (a, d) −EAMT) for connected graphs has been extensively investigated. For instances, in Baˇca et al. (2007) showed that wheel W_n has a (a, d) −SEAMT labeling if and only if $d = 1$ and $n \equiv 1 (\mathrm{mod}\ 4)$; Fan F_n has a (a,d)-SEAMT $if\ 2 \leq n \leq 6\ and\ d \in \{0, 1, 2\}$. Ngurah and Baskoro (2003) proved that for every Petersen graph $p(n, m), n \geq 3, 1 \leq m \leq \frac{n}{2}$, has a $(4n + 2, 1)$-SEAMT labeling. More results concerning anti magic total labeling, see for instances (Simanjuntak et al., 2000; Baˇca et al., 2001) Further results in this direction can be found in the comprehensive survey from a nice paper by Gallian (2024).

People also consider how to construct a new (bigger) (a, 0)-SEAMT graphs from some known (smaller) (a, 0)-SEAMT graphs. These constructions are proposed by inserting some new pendant edges and points, see for instance (Sudarsana et al., 2005; Baskoro et al., 2005; Sudarsana et al., 2009; Sudarsana et al., 2021). Other variations and extensions of labeling schemes including generalized antimagic labelings and related structures have been investigated in (Sabaini et al., 2023; Baˇca et al. (2003); Ponraj and Prabhu, (2024) and Rahmadani et al. (2025), thereby enriching the landscape of labeling theory with diverse perspectives and techniques.

In this paper, we establish a construction of (a, d)-SEAMT labelings for certain unions of graphs, in particular for $2P_{2n} \cup S_{n+1,n-1}$ with $n \geq 3$ and $d \in \{0, 1, 2\}$; as well as for $C_n \cup 2K_{1,\frac{n}{2}-2}$, with even $n \geq 6$ and $d \in \{0, 2\}$.

2. Research Methods

2.1 Basic Counting on EAMT (EMT) and SEAMT (SEMT) Labeling

For any (*a, d*)-SEAMT labeling on a (p, q) −graph *G*, the maximum edge-weight is no more than $p + (p - 1) + (p + q)$. Thus,

$$a + (q - 1)d \leq 3p + q - 1. \tag{1}$$

Similarly, the minimum possible edge-weight is at least $1 + 2 + p + 1$. This implies that

$$a \geq p + 4. \tag{2}$$

So, from (1) and (2), we have

$$d \leq \frac{2p + q - 5}{q - 1}. \tag{3}$$

In general, for any (a, d) −EAMT labeling on a (p, q) −graph *G*, the maximum edge-weight is no more than $(p + q - 2) + (p + q - 1) + (p + q)$. Thus,

$$a + (q - 1)d \leq 3p + 3q - 3 \tag{4}$$

Similarly, the minimum possible edge-weight is at least $1 + 2 + 3$. Consequently,

$$a \geq 6 \tag{5}$$

And from (4) and (5), we have

$$d \leq \frac{3p + 3q - 5}{q - 1} \tag{6}$$

Let d_i be the degree of vertices v_i with $i = 1,2,\ldots p$. Furthermore, based on elementary counting on the EAMT graph, we obtain that:

$$qa + d\frac{q(q-1)}{2} = \frac{(p+q)(p+q+1)}{2} + \sum_{i=1}^{p}(d_i - 1)f(v_i) \tag{7}$$

Meanwhile, an (*a, d*)-EAMT (SEAMT) graph is an EMT (SEMT) graph with the magic constant *a*, if $d = 0$, in this case we replace symbol of the magic constant *a* by *k*. Then, we have the following equation:

$$qk = \frac{(p+q)(p+q+1)}{2} + \sum_{i=1}^{p}(d_i - 1)f(v_i) \tag{8}$$

Therefore, if q is even, $p + q \equiv 2 \pmod 4$, and every d_i is odd, then Equation (8) cannot be satisfied. Consequently, if a graph *G* of order *p* and size *q* satisfies these conditions then *G* is not an EMT (SEMT) graph.

Equation (8) can be used to provide bounds of the magic constant *k* of EMT graph *G*. Let $d_1 \leq d_2 \leq \ldots \leq d_p$, we have that:

$$\left\lceil \frac{(p+q)(p+q+1) + 2\sum_{i=1}^{p}(d_i - 1)(p+1-i)}{2q} \right\rceil \leq k \leq \left\lfloor \frac{(p+q)(p+q+1) + 2\sum_{i=1}^{p}(d_i - 1)(q+i)}{2q} \right\rfloor \tag{9}$$

Equation (8) also can be used to provide bounds of the magic constant *k* for SEMT graph *G*. Based on Equation (9), we obtain that:

$$\left\lceil \frac{(p+q)(p+q+1) + 2\sum_{i=1}^{p}(d_i - 1)(p+1-i)}{2q} \right\rceil \leq k \leq \left\lfloor \frac{(p+q)(p+q+1) + 2\sum_{i=1}^{p}(d_i - 1)(i)}{2q} \right\rfloor \tag{10}$$

2.2 The Duality of EAMT and SEAMT Labeling

Given any (*a, d*)-EAMT labeling λ on a (p, q) −graph *G*. Then, its dual labeling λ' can be defined Sugeng and Miller (2005) by $\lambda(x) = p + q + 1 - \lambda(x)$, for any vertex x, and $\lambda(xy) = p + q + 1 - \lambda(xy)$, for any edge xy. By using this above duality property, we have the following theorem.

Theorem A. [Sugeng and Miller (2005)] *If a (p, q)-graph G has an (a, d)-EAMT labeling then G has an* $(3p + 3q + 3 - a - (q - 1)d, d)$*-EAMT labeling as its dual.*

Theorem B. [Sudarsana et al. (2009)] *Let* λ_1 *be a (a, d)-SEAMT labeling of a* (p, q) *−graph G. Then, the labeling* λ'_1 *defined:*

$\lambda'_1(x) = p + 1 - \lambda_1(x), \forall x \in V$, *and*

$\lambda_1(xy) = 2p + q + 1 - \lambda 1(xy), \forall xy \in E$

is a $(4p + q + 3 - a - (q - 1)d, d)$*-SEAMT labeling of G.*

The labeling λ'_1 is called a *dual* (a, d) *−SEAMT labeling of* λ_1 on G.

2.3 Some Lemmas

The properties of (*a, d*)-SEAMT graph proposed in the next lemmas will be useful in the next section. Baˇca et al., (2003) and Sudarsana et al. (2009) have proved the following lemmas.

Lemma 1. [Baˇca et al. (2003)] *If a (p, q)-graph G has a bijection* $f: V(G) \rightarrow \{1, 2, \dots, p\}$ *such that the set* $S_f = \{f(u) + f(v) : uv \in E(G)\}$ *consists of q consecutive integers with difference d then G has a* $(a_1, d - 1)$ *−SEAMT and a* $(a_2, d+1)$*-SEAMT labeling with* $a_1 = min(S) + p + q$ *and* $a_2 = min(S) + p + 1$.

Lemma 2. [Sudarsana et al., (2009)] *Let G be a (p, q)-graph having a bijection* $f: V(G) \rightarrow \{1,2, \dots, p\}$ *such that the set* $S_f = \{f(u) + f(v) : uv \in E(G)\}$ *consists of q consecutive integers. If q is odd then G has a (a, 1)-SEAMT labeling with* $a = min(S_f') + p + \frac{q+1}{2}$ *or* $a = min(S_f') + p + \frac{q+1}{2} + 1$.

3. Results and Discussion

3.1 On the union of some graphs

The theorems established in this section concern the existence of (a, d) −SEAMT labelings for the graphs $2P_{2n} \cup S_{n+1,n-1}$ and $C_n \cup 2K_{1,\frac{n}{2}-2}$ with parameter $d \in \{0, 2\}$.

Theorem 1. *The graph* $2P_{2n} \cup S_{n+1,n-1}$ *has a SEMT labeling with the magic constant* $k = 15n + 1$, *for* $n \geq 3$

Proof. We denote the graph $2P_{2n} \cup S_{n+1,n-1}$ having the vertex set and the edge set as follows:

$$V(2P_{2n} \cup S_{n+1,n-1}) = \{v_{1,j} | 1 \leq i \leq 3, 1 \leq j \leq n\} \quad (11)$$

$$E(2P_{2n} \cup S_{n+1,n-1}) = \{e_{1,j} | i = 1,3; 1 \leq j \leq 2n - 1\} \cup \{e_{1,j} | i = 2; 1 \leq j \leq 2n - 1\} \quad (12)$$

with $e_{1,j} = v_1, j\, v_{1,j}; 1 \leq j \leq 2n - 1,$

$$e_{2,j} = \begin{cases} v_{2,n+2}v_{2,j}, 1 \leq j \leq n+1 \\ v_{2,n+1}v_{2,j+1}; n+2 \leq j \leq 2n-1 \end{cases}$$

$e_{3,j} = v_{3,j}v_{3,j+1}; n \leq 2n - 1.$

Labels the vertex set and the adge sets as follows:

$$\lambda(v_{i,j}) = \begin{cases} \frac{3j+1}{2}; i = 1, 1 \leq j \leq 2n-1 \text{ and odd } j \\ 3n + \frac{3j}{2} - 1; i = 1, 2 \leq 2n-1, \quad \text{and even } j \\ \frac{3j+3}{2}; i = 3, \quad 1 \leq j \leq 2n-1, \text{and odd } j \\ 3n + \frac{3j}{2}; i = 3, 2 \leq j \leq 2n, \text{and even } j. \end{cases} \quad (13)$$

$$(v_{i,j}) = \begin{cases} 12n-3j;\ i=1, 1\le j\le 2n-1 \\ 12n-3j-1;\ i=2, 1\le j\le 2n-1 \\ 12n-3j-2;\ i=3, 1\le j\le 2n-1 \end{cases} \tag{14}$$

Therefore, the magic constant of graph $2P_{2n} \cup S_{n+1,n-1}$ can be determined by:

$$k = \begin{cases} \lambda(v_{1,j}) + \lambda(e_{1,j}) + \lambda(v_{1,j+1}) \\ \lambda(v_{2,n+2}) + \lambda(e_{2,j}) + \lambda(v_{2,j}) = \lambda(v_{2,n+1}) + \lambda(e_{2,j}) + \lambda(v_{2,j+1}) \\ \lambda(v_{3,j}) + \lambda(e_{3,j}) + \lambda(v_{3,j+1}) \end{cases} \tag{15}$$

$$k = \begin{cases} \frac{3j+1}{2} + (12n-3j) + 3n + \frac{3(j+1)}{2} - 1 = 15n+1 \\ 3(n+2)-2) + (12-3j-1) + 3((j)-2) = 15n+1 \\ \frac{3j+3}{2} + (12n-3j-2) + (3n + \frac{3(j+1)}{2}) = 15n+1 \end{cases} \tag{16}$$

Thus, the graph $2P_{2n} \cup S_{n+1,\ n-1}$ has a SEMT labeling with the magic constant $k = 15n + 1$, for $n \ge 3$. ■

Theorem 2. *The graph* $C_n \cup 2K_{1,\frac{n}{2}-2}$ *has a SEMT labeling with the magic constant* $k = 5n - 5$ *for even* $n \ge 6$.

Proof. The vertex set and the edge set of the graph $C_n \cup 2K_{1,\frac{n}{2}-2}$ for even $n \ge 6$ can be denoted respectively by the following notation

$$V(C_n \cup 2K_{1,\frac{n}{2}-2}) = \{v_{i,j} | 1 \le j \le n\} \cup v_{i,j}\ |2 \le i \le 3, 1 \le j \le \frac{n}{2} - 2 \tag{17}$$

$$E(C_n \cup 2K_{1,\frac{n}{2}-2}) = \{e_{i,j} | 1 \le j \le n\} \cup e_{i,j}\ |2 \le i \le 3, 1 \le j \le \frac{n}{2} - 2\}, \tag{18}$$

with

$$e_{1,j} = \begin{cases} v_{1,j}v_{1,j+1};\ 1 \le j \le n-1 \\ v_{1,}1v_{1,j};\ j = n \end{cases}$$

$$e_{i,j} = v_{i,}1v_{i,j};\ 2 \le i \le 3, 2 \le \frac{n}{2} - 2.$$

Labels the vertex set and the edge sets in the following:

$$\lambda(v_{i,j}) = \begin{cases} \frac{j+1}{2};\ i = 1, 1 \le j \le n-1, \text{and odd } j \\ \frac{2n+j}{2} - 1;\ i = 1, 2 \le j \le n-2, \text{and even } j \\ n+j-2;\ i = 1, j = n \\ \frac{n+j+1}{2};\ i = 2, j = 1 \\ \frac{3n+2j}{2} - 2;\ i = 2, 2 \le j \le \frac{n}{1} - 2 \\ 2n-j-2;\ i = 3,\ j = 1 \\ \frac{n+2}{2} + j;\ i = 3, 2 \le j \le \frac{n}{2} - 2 \end{cases} \tag{19}$$

$$\lambda(e_{i,j}) = \begin{cases} 4n-5-j; i=1, j=n \\ 2j-\frac{n}{2}-3; i=1, j=n \\ 3n-4-j; i=2, 1 \le j \le \frac{n}{2}-2 \\ \frac{5n}{2}-j-3; i=3, 1 \le j \le \frac{n}{2}-2. \end{cases} \quad (20)$$

Furthermore, we obtain that

$$k = \begin{cases} \lambda(v_{i,j}) + \lambda(e_{i,j}) + \lambda(v_{i,j+1}), i=1, 1 \le j \le n-2 \\ \lambda(v_{1,1}) + \lambda(e_{1,j-1}) + \lambda(v_{i,j}), i=1, j=n \\ \lambda(v_{2,1}) + \lambda(e_{2,j-1}) + \lambda(v_{i,j}), i=2, 1 \le j \le \frac{n}{2}-2 \\ \lambda(v_{3,1}) + \lambda(e_{3,j}) + \lambda(v_{3,j+1}), i=3, 1 \le j \le \frac{n}{2}-2 \end{cases} \quad (21)$$

$$k = \begin{cases} \frac{j+1}{2}+4n-5-j+\frac{2n+(j+1)}{2}-1 = \frac{10n-10}{2} = 5n-5 \\ \frac{1+1}{2}+2+4n-5-(j-1)+n+j-2 = \frac{10n-10}{2} = 5n-5 \\ \frac{n+j+1}{2}+3n-4-5(j-1)+n+j-2 = \frac{10n-10}{2} = 5n-5 \\ 2n-j-2+\frac{5n}{2}-j-3+\frac{n+2}{2}+j = \frac{10n-10}{2} = 5n-5 \end{cases} \quad (22)$$

Thus, it is shown that the graph $C_n \cup 2K_{1,\frac{n}{2}-2}$ has a SEMT labeling for even $n \ge 6$ with the magic constant $k = 5n-5$. ■

Based on **Theorems 1** and **2**, it has been confirmed that the graphs $2P_{2n} \cup S_{n+1,n-1}$ and $C_n \cup 2K_{1,\frac{n}{2}-2}$ admit SEAMT labelings with magic constants $k = 15n+1, 5n-5$ and $d = 0$. Building upon these results, we extend the discussion by demonstrating that the graph $2P_{2n} \cup S_{n+1,n-1}$ possesses an (a,d) −SEAMT labeling, for all integers $n \ge 3$ with $d \in \{0,1,2\}$. In addition, it is also established that the graph $C_n \cup 2K_{1,\frac{n}{2}-2}$ admits an (a, d)-SEAMT labeling whenever n is even with $n \ge 6$ and $d \in \{0, 2\}$. These findings provide further evidence of the applicability of (a, d)-SEAMT labelings to a wider class of graph unions, thereby enriching the existing body of knowledge in the study of graph labelings, as will be formally stated and proven in **Theorems 3** and **4** below.

Theorem 3. *For every integer $n \ge 3$, the graph $2P_{2n} \cup S_{n+1,n-1}$ admits three distinct (a, d)-SEAMT labelings, namely a $(15n+1, 0)$-SEAMT labeling, a $(12n+3, 1)$-SEAMT labeling, and a $(9n+5, 2)$-SEAMT labeling.*

Proof. We begin by defining a vertex bijection $\lambda: V(2P_{2n} \cup S_{n+1,n-1}) \to \{1,2, \dots, 12n-3\}$ for every integer $n \ge 3$ where the construction of λ is carried out in accordance with the method previously introduced in the proof of **Theorem 1**. The explicit form of this bijection guarantees that each vertex of the graph $2P_{2n} \cup S_{n+1,n-1}$is uniquely assigned a distinct integer label within the prescribed range. By invoking **Lemma 1**, it follows directly that the labeling obtained from λ can be naturally extended to yield both a $(15n+1, 0)$ SEAMT labeling and a $(9n+5, 2)$-SEAMT labeling of the graph under consideration, valid for all integers $n \ge 3$. Moreover, it is important to observe that the cardinality of the edge set of $2P_{2n} \cup S_{n+1,n-1}$ is given by $q = 6n-3$ which is an odd integer for every $n \ge 3$. This structural property of the graph plays a crucial role in the application of **Lemma 2,** since the lemma ensures that whenever the number of edges is odd, the bijection λ can be further extended to produce a new family of labelings. Consequently, for even values of $n \ge 2$, the construction described above also leads to the existence of a $(12n+3, 1)$ SEAMT labeling of the graph $2P_{2n} \cup S_{n+1,n-1}$, thereby demonstrating the versatility of the labeling method and highlighting the interplay between vertex bijections, edge cardinality, and the applicability of the supporting lemmas. ■

Theorem 4. *For every even integer $n \geq 6$, the graph $C_n \cup 2K_{1,\frac{n}{2}-2}$ admits both a $(5n-5, 0)$ −SEAMT labeling and a $(3n, 2)$ −SEAMT labeling.*

Proof. We begin by defining a vertex bijection $\lambda: V(C_n \cup 2K_{1,\frac{n}{2}-2}) \to \{1, 2, \dots, 4n-6\}$, for every even integer $n \geq 6$. The construction of this bijection is carried out in accordance with the method introduced in the proof of **Theorem 2**, thereby ensuring that each vertex of the graph $C_n \cup 2K_{1,\frac{n}{2}-2}$ is uniquely assigned to a distinct integer within the specified range. Once the bijection λ has been established, we proceed by invoking **Lemma 1**. The lemma guarantees that under the given construction, the labeling induced by λ can be naturally extended to yield valid (a, d)-SEAMT labeling. In particular, a careful verification shows that the bijection λ extends to both a $(5n-5, 0)$ −SEAMT labeling and a $(3n, 2)$ −SEAMT labeling of graph $C_n \cup 2K_{1,\frac{n}{2}-2}$.

Therefore, for every even integer $n \geq 6$ the graph under consideration admits two distinct families of (a, d)-SEAMT labelings, which completes the proof.■

By combining the results established in **Theorems A** and **B** with the structural property guaranteed by **Lemma 2**, we are able to derive a number of immediate consequences that further clarify the labeling behavior of the graphs under consideration. These consequences are formulated in the following corollaries, which serve both as direct extensions of the preceding theorems and as additional evidence of the applicability of (a, d)-SEAMT labelings to broader classes of graphs.

Corollary 1. *For every even integer $n \geq 3$, the graph $2P_{2n} \cup S_{n+1,n-1}$ admits both a $(21n-7, 0)$ −EAMT labeling and a $(15n-3, 2)$ −EAMT labeling.*

Corollary 2. *For every integer $n \geq 3$, the graph $2P_{2n} \cup S_{n+1,n-1}$ admits both a $(15n-1, 0)$ −SEAMT labeling and a $(9n+3, 2)$ −SEAMT labeling. Moreover, the graph $2P_{2n} \cup S_{n+1,n-1}$ also admits a $(12n+1, 1)$ −SEAMT labeling for all integers $n \geq 3$.*

Corollary 3. *For every even integer $n \geq 6$, the graph $C_n \cup 2K_{1,\frac{n}{2}-2}$ admits both a $(7n-10, 0)$ −EAMT labeling and a $(5n-5, 2)$ −EAMT labeling.*

Corollary 4. *For every even integer $n \geq 6$, the graph $C_n \cup 2K_{1,\frac{n}{2}-2}$ admits both a $(5n-4, 0)$ −SEAMT labeling and a $(3n+1, 2)$ −SEAMT labeling.*

After presenting the preceding theorems and corollaries, which establish the existence and construction of various (a, d)-SEMT and EAMT labelings for different classes of graphs, it is essential to turn our focus toward a deeper structural aspect of such labelings. In particular, we now investigate the uniqueness of the magic constant in SEMT labelings of regular graphs, a property that not only strengthens the results already obtained but also provides a more comprehensive understanding of the inherent regularities within this labeling framework.

3.2 The Uniqueness of Magic Constant of SEMT in Regular Graph

Theorem 5. *Let G be a SEMT graph with order p and size q. If G is regular graph with degree r then the magic constant k of G is unique, that is* $k = \left\lfloor \frac{(r+2)(2p+pr+2)+4(r-1)(p+1)}{4r} \right\rfloor$.

Proof. Let G be a SEMT regular graph of degree r, then the magic constant of G exists, say that k. Based on the Equation (10) we have:

$$\left\lceil \frac{(p+q)(p+q+1)+2\sum_{i=1}^{p}(r-1)(p+1-i)}{2q} \right\rceil \le k \le \left\lfloor \frac{(p+q)(p+q+1)+2\sum_{i=1}^{p}(r-1)(i)}{2q} \right\rfloor$$

It can be verified that $\sum_{i=1}^{p}(r-1)(p+1-i) = \sum_{i=1}^{p}(r-1)(i)$ and then

$$\left\lceil \frac{(p+q)(p+q+1)+2\sum_{i=1}^{p}(r-1)(i)}{2q} \right\rceil \le k \le \left\lfloor \frac{(p+q)(p+q+1)+2\sum_{i=1}^{p}(r-1)(i)}{2q} \right\rfloor$$

Based on squeeze principle we obtain that

$$k = \left\lceil \frac{(p+q)(p+q+1)+2\sum_{i=1}^{p}(r-1)(i)}{2q} \right\rceil = \left\lfloor \frac{(p+q)(p+q+1)+2(r-1)\sum_{i=1}^{p} i}{2q} \right\rfloor$$

$$k = \left\lfloor \frac{(p+q)(p+q+1)+2(r-1)\sum_{i=1}^{p} i}{2q} \right\rfloor = \left\lfloor \frac{(p+q)(p+q+1)+(r-1)p(p+1)}{2q} \right\rfloor$$

Recall that $q = \frac{pr}{2}$, then

$$k = \left\lfloor \frac{\left(p+\frac{pr}{2}\right)\left(p+\frac{pr}{2}+1\right)+(r-1)p(p+1)}{pr} \right\rfloor$$

$$= \left\lfloor \frac{(r+2)(2p+pr+2)+4(r-1)(p+1)}{4r} \right\rfloor$$

This concludes the proof of theorem. ■

3.3 Discussion

We investigate finite undirected graphs devoid of loops and multiple edges, focusing on the concept of (a, d)-edge anti magic total (EAMT) labeling. Specifically, a $(p, q)-$graph G is termed $(a, d)-$EAMT if there exists a bijection $\lambda: V\ E \rightarrow \{1, 2, \cdots, p+q\}$ that assigns to each edge xy a weight $w(xy) = (x) + (y) + (xy)$ such that the set of edge-weights forms an arithmetic progression starting from a with common difference d. If the bijection satisfies $\lambda(V) = \{1, 2, \cdots, p\}$, the labeling λ is called a super $(a, d)-$EAMT ((a, d)-SEAMT). We establish bounds on the parameters a and d based on vertex degrees and the total number of vertices and edges, and we explore the duality properties of both EAMT and SEAMT labelings. Building on these foundational results and existing literature, we propose new constructions of $(a, d)-$SEAMT labelings for unions of graphs, particularly for graphs of the form $2P_{2n} \cup S_{n+1,n-1}$ for $n \ge 3$ (with magic constants such as $15n+1, 12n+3$, and $9n+5$ corresponding to different values of d) and $C_n \cup 2K_{1,\frac{n}{2}-2}$ for even $n \ge 6$ (with magic constants $5n-5$ and $3n$ for selected d values)

These constructions not only extend the known classes of antimagic total labeling but also provide insights into the combinatorial structures underlying graph labeling, thereby contributing to both theoretical advancements and practical applications in graph theory.

4. Conclusions

In this paper, we have explored the properties and constructions of (a, d)-edge anti magic total (EAMT) and super (a, d) −EAMT (SEAMT) labelings in finite undirected graphs. We established key bounds on the parameters a and d based on graph structure and presented duality properties that link EAMT and SEAMT labelings. Notably, we introduced new constructions for (a, d) −SEAMT labelings on unions of graphs, specifically for $2P_{2n} \cup S_{n+1,n-1}$ (with labelings such as $(15n+1, 0)$ −SEAMT, $(12n+3, 1)$ −SEAMT, and $(9n+5, 2)$−SEAMT for $n \geq 3$) and $C_n \cup 2K_{1,\frac{n}{2}-2}$(with $(5n-5, 0)$ −SEAMT and $(3n, 2)$−SEAMT labelings for even $n \geq 6$). These results not only extend the current body of knowledge on magic type labelings but also offer a framework for constructing new labelings in broader classes of graphs. The duality theorems and corollaries further reinforce the versatility of our approach, demonstrating that these methods can be applied to derive complementary EAMT labelings. Overall, our contributions provide valuable insights into graph labeling theory, opening avenues for future research in both theoretical developments and practical applications in combinatorial design and network analysis.

Conflicts of interest

The authors declare no competing interests.

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