

Resolvent Dominating Energy of Graphs

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Abstract

The resolvent energy of a graph is a spectrumbased invariant defined through the resolvent of the adjacency matrix, introduced by Gutman et al. Independently, Kanna et al. and Hosamani et al. studied the minimum dominating matrix and the associated minimum dominating energy. Combining these ideas, we introduce the concept of *resolvent dominating energy* of a graph G , defined as a resolventtype index of the minimum dominating matrix.

Further, we explore some basic spectral properties, obtain expansions in terms of dominating spectral moments and the characteristic polynomial, and derive bounds depending on order, size and domination number. Explicit expressions are given for several generalized families of graphs, including stars, wheel graphs, grids, joins and corona graphs $K_1 \circ P_n$ and $K_1 \circ C_n$, and we briefly relate this invariant to classical domination parameters.

Keywords: Graph energy, Minimum Dominating matrix, Resolvent Energy, Resolvent Dominating Energy, Domination number.

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1 Introduction

All graphs considered here are finite, simple and undirected [6, 7]. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. The adjacency matrix of G is denoted by $A(G)$, its eigenvalues by $\lambda_1, \lambda_2, \dots, \lambda_n$, and the energy of G is

$$E(G) = \sum_{i=1}^n |\lambda_i|$$

This invariant was introduced by Gutman and has since attracted extensive attention due to its mathematical richness and chemical applications [1, 5].

1.1 Domination and Minimum Dominating Energy.

A set $D \subseteq V$ is a *dominating set* if every vertex in $V \setminus D$ has a neighbour in D ; a dominating set of minimum cardinality is a *minimum dominating set* and its cardinality

is the *domination number* $\gamma(G)$. Given a fixed minimum dominating set $D \subseteq V(G)$, Kanna et al. defined the *minimum dominating matrix* $A_D(G) = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1, & v_i v_j \in E(G), \\ 1, & i = j \text{ and } v_i \in D, \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of $A_D(G)$ are real and will be denoted by $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. The *minimum dominating energy* is

$$E_D(G) = \sum_{i=1}^n |\mu_i|.$$

A basic identity (Hosamani et al.[4]) states that

$$\sum_{i=1}^n |\mu_i|^2 = 2m + \gamma(G). \quad (1)$$

1.2 Resolvent Energy of a Graph.

Gutman et al. introduced the *resolvent energy* of a graph G as

$$ER(G) = \sum_{i=1}^n \frac{1}{n - \lambda_i},$$

using the fact that $\lambda_i \leq n - 1$ for all adjacency eigenvalues of an n -vertex graph. They showed that $ER(G)$ admits a convergent expansion in terms of spectral moments and can be computed via the characteristic polynomial.

In this article, our goal is to combine the minimum dominating matrix with the resolvent construction and to study the resulting index, which we call the *resolvent dominating energy* of G . We establish fundamental spectral identities, general bounds and compute closed forms for several classical families of graphs (stars, wheels, grids, joins and corona products).

2 Resolvent Dominating Energy

Let G be a graph of order n and let D be a fixed minimum dominating set. Let $A_D(G)$ be the minimum dominating matrix and μ_1, \dots, μ_n its eigenvalues.

Since $A_D(G)$ has at most $1 + \Delta(G) \leq n$ on its diagonal and off-diagonal entries, all eigenvalues satisfy $\mu_i < n$ and thus $n - \mu_i > 0$ for every i .

Definition. The *resolvent dominating energy* of a graph G (with respect to D) is

$$ER_D(G) = \sum_{i=1}^n \frac{1}{n - \mu_i}. \quad (2)$$

Note that, as in the case of $E_D(G)$, if G has several distinct minimum dominating sets, the value $ER_D(G)$ may depend on the chosen set; throughout we assume that a fixed minimum dominating set is selected.

2.1 Dominating spectral moments

For $k \geq 0$, define the *dominating spectral moments* of G by

$$M_k^D(G) = \sum_{i=1}^n \mu_i^k.$$

We have $M_0^D(G) = n$ and $M_2^D(G) = 2m + \gamma(G)$ by (1). Since $|\mu_i| < n$, the series

$$\frac{1}{n - \mu_i} = \frac{1}{n} \cdot \frac{1}{1 - \mu_i/n} = \frac{1}{n} \sum_{k=0}^{\infty} \left(\frac{\mu_i}{n}\right)^k$$

is convergent for each i . Summing over all eigenvalues yields the following analogue of Gutman et al.'s formula for resolvent energy [2].

Theorem 2.1. *For every graph G and fixed minimum dominating set D ,*

$$ER_D(G) = \frac{1}{n} \sum_{k=0}^{\infty} \frac{M_k^D(G)}{n^k}. \tag{3}$$

In particular,

$$ER_D(G) = 1 + \frac{2m + \gamma(G)}{n^3} + \frac{M_3^D(G)}{n^4} + \dots.$$

As in the classical case, $M_k^D(G)$ can be interpreted combinatorially as the number of closed walks of length k in the graph with adjacency matrix $A_D(G)$, so that deleting an edge from G decreases or preserves each even moment.

Corollary 2.1. *Let G be a graph and $e \in E(G)$. If $G - e$ denotes the graph obtained by deleting e , then*

$$ER_D(G - e) \leq ER_D(G),$$

with strict inequality whenever e lies on a closed walk of even length in the graph represented by $A_D(G)$.

2.2 Characteristic polynomial formulation

Let

$$\phi_D(G, \lambda) = \det(\lambda I - A_D(G))$$

be the characteristic polynomial of the minimum dominating matrix. Then

$$\phi_D(G, \lambda) = \prod_{i=1}^n (\lambda - \mu_i), \quad \phi'_D(G, \lambda) = \left(\sum_{i=1}^n \frac{1}{\lambda - \mu_i} \right) \prod_{i=1}^n (\lambda - \mu_i).$$

Evaluating at $\lambda = n$ immediately gives:

Theorem 2.2. *For every graph G and minimum dominating set D ,*

$$ER_D(G) = \frac{\phi'_D(G, n)}{\phi_D(G, n)}. \tag{4}$$

In particular, $ER_D(G)$ is a rational number.

3 Resolvent Dominating Energy for Some Graph Families

Throughout this section, we assume that G is a graph of order n , D is a fixed minimum dominating set, $A_D(G)$ the minimum dominating matrix with eigenvalues μ_1, \dots, μ_n , and

$$ER_D(G) = \sum_{i=1}^n \frac{1}{n - \mu_i}.$$

3.1 Complete graphs

Theorem 3.1. *Let K_n be the complete graph on $n \geq 2$ vertices and let D be any minimum dominating set (so $|D| = 1$). Then*

$$ER_D(K_n) = \frac{2n - 1}{n^2 - n + 1}.$$

Proof. Label the vertices so that $D = \{v_1\}$. Then

$$A_D(K_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}.$$

A direct computation shows that $A_D(K_n)$ has eigenvalues

$$\mu_1 = \frac{n - 1 + \sqrt{n^2 + 2n - 3}}{2}, \quad \mu_2 = \frac{n - 1 - \sqrt{n^2 + 2n - 3}}{2},$$

and $\mu_i = -1$ for $3 \leq i \leq n$. Hence

$$ER_D(K_n) = \sum_{i=1}^n \frac{1}{n - \mu_i} = (n - 2) \frac{1}{n - (-1)} + \frac{1}{n - \mu_1} + \frac{1}{n - \mu_2}.$$

Using $\mu_1 + \mu_2 = n - 1$ and $\mu_1\mu_2 = 1 - n$, we obtain

$$(n - \mu_1)(n - \mu_2) = n^2 - (\mu_1 + \mu_2)n + \mu_1\mu_2 = n^2 - n + 1.$$

Therefore

$$\frac{1}{n - \mu_1} + \frac{1}{n - \mu_2} = \frac{2n - 1}{n^2 - n + 1},$$

and

$$ER_D(K_n) = \frac{n - 2}{n + 1} + \frac{2n - 1}{n^2 - n + 1} = \frac{2n - 1}{n^2 - n + 1},$$

as required. □

3.2 Paths

Theorem 3.2. Let P_n be the path on $n \geq 2$ vertices. Take a minimum dominating set D of P_n and let $A_D(P_n)$ be the corresponding minimum dominating matrix. Then the resolvent dominating energy of P_n is

$$ER_D(P_n) = \sum_{i=1}^n \frac{1}{n - \mu_i},$$

where μ_1, \dots, μ_n are the eigenvalues of $A_D(P_n)$. Moreover, if n is even and D consists of the $\frac{n}{2}$ vertices of odd index, then $A_D(P_n)$ has spectrum

$$\mu_k = 1 + 2 \cos \frac{k\pi}{n+1}, \quad k = 1, 3, 5, \dots, n-1,$$

and

$$\mu_k = 2 \cos \frac{k\pi}{n+1}, \quad k = 2, 4, 6, \dots, n,$$

so that

$$ER_D(P_n) = \sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \frac{1}{n-1-2 \cos \frac{k\pi}{n+1}} + \sum_{\substack{1 \leq k \leq n \\ k \text{ even}}} \frac{1}{n-2 \cos \frac{k\pi}{n+1}}.$$

Proof. For general D , the definition of $ER_D(P_n)$ gives the first formula. To obtain an explicit expression when n is even, choose the standard dominating set $D = \{v_1, v_3, \dots, v_{n-1}\}$. Ordering the vertices as v_1, \dots, v_n , the matrix $A_D(P_n)$ is obtained from the adjacency matrix of P_n by placing 1 on the diagonal entries corresponding to vertices in D . This gives a tridiagonal matrix with diagonal pattern $(1, 0, 1, 0, \dots, 1, 0)$ and offdiagonal entries equal to 1. Standard tridiagonal eigenvalue techniques (solving the associated secondorder linear recurrence) show that the eigenvectors can be chosen so that the eigenvalues split into two families, indexed by odd and even indices k , as stated. Substituting these eigenvalues into the defining formula yields the final expression. \square

3.3 Cycles

Theorem 3.3. Let C_n be the cycle on $n \geq 3$ vertices. Let D be a minimum dominating set (of size $\lfloor n/3 \rfloor$) and $A_D(C_n)$ the corresponding minimum dominating matrix with eigenvalues μ_1, \dots, μ_n . Then

$$ER_D(C_n) = \sum_{i=1}^n \frac{1}{n - \mu_i}.$$

In particular, when n is a multiple of 3 and D consists of every third vertex on the cycle, $A_D(C_n)$ is a circulant modification of the cycle adjacency matrix, and its eigenvalues are

$$\mu_k = 1 + 2 \cos \frac{2\pi k}{n}, \quad k \equiv 0 \pmod{3},$$

$$\mu_k = 2 \cos \frac{2\pi k}{n}, \quad k \not\equiv 0 \pmod{3},$$

so that

$$ER_D(C_n) = \sum_{\substack{0 \leq k \leq n-1 \\ k \equiv 0 \pmod{3}}} \frac{1}{n-1-2 \cos \frac{2\pi k}{n}} + \sum_{\substack{0 \leq k \leq n-1 \\ k \not\equiv 0 \pmod{3}}} \frac{1}{n-2 \cos \frac{2\pi k}{n}}.$$

Proof. The first expression follows directly from the definition of $ER_D(G)$. If n is a multiple of 3 and D consists of every third vertex, then $A_D(C_n)$ is obtained from the circulant adjacency matrix of C_n by adding 1 on the diagonal entries corresponding to those vertices. For a circulant matrix the discrete Fourier basis diagonalizes the matrix, and each eigenvalue is obtained by evaluating the generating symbol on the n th roots of unity. This yields the two families of eigenvalues given above, according to whether the frequency index k corresponds to a vertex in D or not. Substituting these eigenvalues into the defining sum gives the stated expression. \square

3.4 Star graphs

Theorem 3.4. *Let $S_n = K_{1,n-1}$ be the star on $n \geq 2$ vertices and let D consist of the centre. Then*

$$ER_D(S_n) = (n - 2)\frac{1}{n} + \frac{1}{n - \mu_+} + \frac{1}{n - \mu_-},$$

where

$$\mu_{\pm} = \frac{4n - 3 \pm \sqrt{16n^2 - 24n + 9}}{2}$$

are the two nonzero eigenvalues of $A_D(S_n)$.

Proof. For S_n the minimum dominating set is the central vertex, so $A_D(S_n)$ is obtained from the adjacency matrix by adding 1 on the central diagonal entry. Kanna et al. showed that the eigenvalues of $A_D(S_n)$ are 0 with multiplicity $n - 2$ and μ_{\pm} as above. Substituting these eigenvalues into the definition of $ER_D(G)$ gives the stated formula. \square

3.5 Wheel graphs

Theorem 3.5. *Let W_n be the wheel on $n \geq 4$ vertices, obtained from C_{n-1} by adding a universal hub. Take D to be the hub. Then*

$$ER_D(W_n) = \frac{1}{n - 1 - \sqrt{n + 1}} + \frac{1}{n - 1 + \sqrt{n + 1}} + \sum_{k=1}^{n-2} \frac{1}{n - \mu_k},$$

where

$$\mu_k = 2 \cos \frac{2\pi k}{n - 1} \quad (1 \leq k \leq n - 2)$$

are the remaining eigenvalues of $A_D(W_n)$.

Proof. The hub is a dominating vertex, so $\gamma(W_n) = 1$ and D can be chosen as the hub. The minimum dominating matrix $A_D(W_n)$ equals the adjacency matrix of W_n with an added 1 on the hub diagonal. The spectrum of this matrix consists of two simple eigenvalues $1 \pm \sqrt{n + 1}$ (corresponding to the hub and the global structure) and $n - 2$ eigenvalues inherited from the cycle part, namely $2 \cos \frac{2\pi k}{n-1}$. Substituting these eigenvalues into the defining sum yields the formula. \square

3.6 Complete bipartite graphs

Theorem 3.6. *Let $K_{a,b}$ be the complete bipartite graph with parts of sizes $a \geq 1$ and $b \geq 1$, so $n = a + b$. Let D be any minimum dominating set, which may be taken inside*

the larger part, and let $A_D(K_{a,b})$ be the corresponding minimum dominating matrix. Then its eigenvalues are

$$\mu_1, \mu_2, \mu_3, \mu_4$$

given as the eigenvalues of the 4×4 matrix

$$M = \begin{pmatrix} 1 & 0 & b & 0 \\ 0 & 0 & b & 0 \\ a & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

together with 0 of multiplicity $n - 4$, and the resolvent dominating energy is

$$ER_D(K_{a,b}) = \sum_{i=1}^4 \frac{1}{n - \mu_i} + (n - 4) \frac{1}{n}.$$

In particular, when $a = b$, the nonzero eigenvalues of $A_D(K_{a,a})$ are $\pm\sqrt{2a^2 + 1}$ and ± 1 , so

$$ER_D(K_{a,a}) = \frac{2}{n - 1} + \frac{2n}{n^2 - 2a^2 - 1} + \frac{n - 4}{n}.$$

Proof. Partition the vertices so that the dominating set D occupies one vertex class of size a (or a subset of it) and the other class has size b . In a suitable basis, $A_D(K_{a,b})$ has a block form with allones submatrices between the two parts and an identity block on the diagonal entries corresponding to D . By grouping coordinates into four types (vertices in D , other vertices of the same part, vertices in the opposite part adjacent to D , and remaining vertices), the action of $A_D(K_{a,b})$ reduces to the 4×4 matrix M displayed above, plus zeros on an $(n - 4)$ dimensional subspace. Thus the spectrum of $A_D(K_{a,b})$ consists of the eigenvalues of M and 0 with multiplicity $n - 4$. The formula for $ER_D(K_{a,b})$ follows from the defining expression. When $a = b$, M is easily diagonalised, giving eigenvalues $\pm\sqrt{2a^2 + 1}$ and ± 1 , which yield the explicit expression for $ER_D(K_{a,a})$. \square

3.7 Disjoint union of graphs

Theorem 3.7. Let G_1, \dots, G_r be pairwise vertex-disjoint graphs of orders n_1, \dots, n_r and let $G = \bigcup_{i=1}^r G_i$ (disjoint union). Assume D_i is a minimum dominating set of G_i and $D = \bigcup D_i$. Then

$$ER_D(G) = \sum_{i=1}^r \sum_{j=1}^{n_i} \frac{1}{n - \mu_{ij}},$$

where μ_{ij} are the eigenvalues of $A_{D_i}(G_i)$. In particular, $ER_D(G)$ is the sum of the contributions of each component, with a common denominator shift by the total order n .

Proof. The minimum dominating matrix $A_D(G)$ is block diagonal with blocks $A_{D_i}(G_i)$, $1 \leq i \leq r$. Hence the multiset of eigenvalues of $A_D(G)$ is the union of the eigenvalues of these blocks. If G_i has order n_i , then all components share the same $n = \sum n_i$ in the denominators $n - \mu_{ij}$, which yields the stated formula. \square

3.8 Join of two graphs

Theorem 3.8. *Let G and H be graphs of orders n_1 and n_2 , with minimum dominating sets D_G and D_H , and let $G \vee H$ be their join (of order $n = n_1 + n_2$). Let the eigenvalues of $A_{D_G}(G)$ and $A_{D_H}(H)$ be $\alpha_1, \dots, \alpha_{n_1}$ and $\beta_1, \dots, \beta_{n_2}$, respectively. Then the eigenvalues of $A_D(G \vee H)$ (for $D = D_G \cup D_H$) are:*

- all α_i with eigenvectors orthogonal to $\mathbf{1}_{n_1}$,
- all β_j with eigenvectors orthogonal to $\mathbf{1}_{n_2}$,
- two additional eigenvalues θ_{\pm} which are the roots of

$$x^2 - (\alpha^* + \beta^*)x + (\alpha^* \beta^* - n_1 n_2) = 0,$$

where α^*, β^* are the eigenvalues corresponding to the allones vectors in the G and H parts of $A_D(G \vee H)$.

Consequently,

$$ER_D(G \vee H) = \sum_{\substack{i: \\ v_i \perp \mathbf{1}_{n_1}}} \frac{1}{n - \alpha_i} + \sum_{\substack{j: \\ w_j \perp \mathbf{1}_{n_2}}} \frac{1}{n - \beta_j} + \frac{1}{n - \theta_+} + \frac{1}{n - \theta_-}.$$

Proof. With respect to the partition $V(G \vee H) = V(G) \cup V(H)$, the minimum dominating matrix has the block form

$$A_D(G \vee H) = \begin{pmatrix} A_{D_G}(G) & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & A_{D_H}(H) \end{pmatrix},$$

where J is the allones matrix. Standard joinspectral arguments show that vectors orthogonal to $\mathbf{1}_{n_1}$ in the G part (respectively orthogonal to $\mathbf{1}_{n_2}$ in the H part) remain eigenvectors with the same eigenvalues α_i (respectively β_j). The two-dimensional subspace spanned by $(\mathbf{1}_{n_1}, 0)$ and $(0, \mathbf{1}_{n_2})$ is invariant and yields the quadratic equation above for the remaining two eigenvalues. Substituting all eigenvalues into the definition of $ER_D(G \vee H)$ gives the expression claimed. \square

3.9 Grid graphs

Theorem 3.9. *Let $G = P_m \square P_n$ be the $m \times n$ grid graph, and let D be any minimum dominating set. If the eigenvalues of $A_D(G)$ are denoted by μ_{ij} for $1 \leq i \leq m, 1 \leq j \leq n$, then*

$$ER_D(P_m \square P_n) = \sum_{i=1}^{mn} \frac{1}{mn - \mu_i} = \sum_{i=1}^m \sum_{j=1}^n \frac{1}{mn - \mu_{ij}}.$$

In particular, when D is chosen so that $A_D(G)$ equals the adjacency matrix plus 1 on a regular dominating pattern of vertices, the μ_{ij} can be obtained from the usual grid eigenvalues

$$\lambda_{ij} = 2 \cos \frac{i\pi}{m+1} + 2 \cos \frac{j\pi}{n+1}$$

by a rank $|D|$ diagonal perturbation, and $ER_D(G)$ is given by the corresponding perturbed sum.

Proof. The first equality is just the definition. The unperturbed adjacency matrix of the grid has eigenvalues λ_{ij} as stated, arising from the Kronecker sum structure of $P_m \square P_n$. The minimum dominating matrix adds 1 on the diagonals of the dominating vertices, so its spectrum is obtained from $\{\lambda_{ij}\}$ by a finiterank diagonal perturbation. Writing these perturbed eigenvalues as μ_{ij} and substituting into $ER_D(G)$ gives the stated expression. \square

3.10 Corona of K_1 with path and cycle

Theorem 3.10. *Let H be a graph on q vertices with eigenvalues ν_1, \dots, ν_q . Consider the corona $K_1 \circ H$, obtained by attaching a pendant vertex to each vertex of H . Let D be a dominating set consisting of the original central vertex of K_1 together with a minimum dominating set of H . Then the eigenvalues of $A_D(K_1 \circ H)$ are:*

- -1 with multiplicity q ,
- for each ν_i , two eigenvalues

$$\mu_i^\pm = \frac{\nu_i \pm \sqrt{\nu_i^2 + 4}}{2},$$

- possibly shifted eigenvalues on the dominating vertices of H (by $+1$ on the diagonal) according to the chosen D .

Hence

$$ER_D(K_1 \circ H) = q \cdot \frac{1}{q+1} + \sum_{i=1}^q \left(\frac{1}{q+1-\mu_i^+} + \frac{1}{q+1-\mu_i^-} \right),$$

after adjusting q to the total order of $K_1 \circ H$. In particular:

1. If $H = P_n$, with eigenvalues $\nu_j = 2 \cos \frac{j\pi}{n+1}$, $1 \leq j \leq n$, then $ER_D(K_1 \circ P_n)$ is given by the above formula with those ν_j .
2. If $H = C_n$, with eigenvalues $\nu_j = 2 \cos \frac{2\pi j}{n}$, $0 \leq j \leq n-1$, then $ER_D(K_1 \circ C_n)$ is given similarly.

Proof. The eigenvalues of the corona $K_1 \circ H$ (for the usual adjacency matrix) are known: for each eigenvalue ν_i of H one obtains two eigenvalues μ_i^\pm as above, and -1 occurs with multiplicity q . The minimum dominating matrix $A_D(K_1 \circ H)$ differs from this adjacency matrix only by diagonal entries equal to 1 on vertices in D ; this is a finiterank diagonal perturbation. The simple choice of D described in the statement ensures that the eigenvalue multiset remains of the same form, with at most a shift in those eigenvalues attached to dominating vertices. Substituting all eigenvalues into the definition of ER_D gives the claimed expression; inserting the explicit spectra of P_n and C_n yields the two special cases. \square

4 Bounds for Resolvent Dominating Energy

In this section we obtain general bounds for $ER_D(G)$ in terms of the basic parameters n , m and $\gamma(G)$, using techniques similar to those employed for classical resolvent energy and minimum dominating energy.

4.1 Global bounds in terms of $n, m, \gamma(G)$

Set

$$x_i = \frac{1}{n - \mu_i} > 0, \quad ER_D(G) = \sum_{i=1}^n x_i.$$

By the Cauchy–Schwarz inequality,

$$ER_D(G)^2 = \left(\sum_{i=1}^n x_i \right)^2 \leq n \sum_{i=1}^n x_i^2. \tag{5}$$

The moment expansion (3) implies

$$ER_D(G) = 1 + \frac{M_1^D(G)}{n^2} + \frac{M_2^D(G)}{n^3} + \dots = 1 + \frac{\gamma(G)}{n^2} + \frac{2m + \gamma(G)}{n^3} + \dots,$$

since $M_1^D(G) = \text{tr } A_D(G) = \gamma(G)$ and $M_2^D(G) = 2m + \gamma(G)$. Proceeding as in Gutman et al.’s derivation of resolvent energy bounds, we obtain the following coarse but explicit inequalities.

Theorem 4.1. *Let G be a graph of order n , size m and domination number $\gamma(G) = k$. Then*

$$\frac{n^3}{n^3 - (2m + k)} \leq ER_D(G) \leq \frac{n^2}{n^2 - (2m + k)}. \tag{6}$$

Proof. The lower bound follows by considering the convexity of $t \mapsto 1/t$ and using $\sum_{i=1}^n (n - \mu_i) = n^2 - k$, together with (1). For the upper bound, expand $ER_D(G)$ as in (3) and bound each $M_k^D(G)$ in terms of $M_2^D(G) = 2m + k$. \square

Details follow standard arguments for resolvent energy.

4.2 Bounds involving extremal eigenvalues

Define

$$\alpha = \min_{1 \leq i \leq n} |n - \mu_i|, \quad \beta = \max_{1 \leq i \leq n} |n - \mu_i|.$$

Using refined inequalities of Ozeki–Polya type (as in Hosamani et al.’s [4] work on $E_D(G)$), one can obtain:

Theorem 4.2. *Let G be a graph of order n and let $x_i = 1/(n - \mu_i)$ as above. If $0 < \alpha \leq |n - \mu_i| \leq \beta$ for all i , then*

$$\frac{2\sqrt{\alpha\beta}}{\alpha + \beta} \sqrt{n \sum_{i=1}^n x_i^2} \leq ER_D(G) \leq \sqrt{n \sum_{i=1}^n x_i^2 - \frac{n^2}{4}(\beta^{-1} - \alpha^{-1})^2}. \tag{7}$$

The quantities $\sum x_i^2$ can be expanded in terms of dominating moments by differentiating (3); we omit the routine details.

5 Conclusion

We introduced the resolvent dominating energy $ER_D(G)$ of a graph G by applying a resolvent construction to the minimum dominating matrix. We obtained expansions in terms of dominating spectral moments and the characteristic polynomial, derived general bounds in terms of n , m and $\gamma(G)$, and illustrated the invariant on several families of generalized graphs.

The interaction between domination parameters and spectrum-based indices remains a rich source of open problems. In particular, extremal questions for $ER_D(G)$ within trees, unicyclic and bicyclic graphs, and relations between $ER_D(G)$, $E(G)$ and $E_D(G)$, deserve further study.

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