A Trek on the Non-homogeneous Quaternary Sextic Surface

$$x^3 + y^3 = 19zw^5$$

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Abstract

The main aim of this paper is to illustrate the process of obtaining non-zero distinct integer solutions to the non-homogeneous quaternary sextic equation given by $x^3 + y^3 = 19zw^5$ Substitution technique and factorization method are utilized to obtain the same. We are able to solve the given sextic equation through the methods suggested above for obtaining plenty of non-zero distinct integer solutions.

Keywords: Non-homogeneous equation, Sextic equation, Quaternary sextic equation, Integer Solutions

Introduction

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Diophantine equations with multi variables [1-41]. This paper aims at determining many integer solutions to non-homogeneous polynomial higher degree equation with four unknowns given by

$x^3 + y^3 = 19 z w^5$. Transformation techniques and factorization methods are utilized to obtain the same.

Method of analysis

The non-homogeneous polynomial equation of degree six with four unknowns under consideration is given by

$$x^3 + y^3 = 19 z w^5 \tag{1}$$

The insertion of the transformation

$$x = u + v, y = u - v, z = 2u, u \neq \pm v \neq 0 \tag{2}$$

in (1) leads to the non-homogeneous ternary heptic equation

$$u^2 + 3v^2 = 19 w^5 \tag{3}$$

To start with, it is seen after some algebra that (3) is satisfied by

$$u = 19^3 r(r^2 + 3s^2)^2, v = 19^3 s(r^2 + 3s^2)^2, w = 19(r^2 + 3s^2)$$

and from (2) ,the corresponding values of x,y,z satisfying (1) are given by

$$x = 19^3 (r + s)(r^2 + 3s^2)^2$$

$$y = 19^3 (r - s)(r^2 + 3s^2)^2$$

$$z = 2*19^3 r(r^2 + 3s^2)^2, r \neq s \neq 0$$

The process of obtaining varieties of integer solutions to (1) through solving (3) is presented below:

Process 1

The choice

$$v = k w^2 \tag{4}$$

in (3) gives

$$u^2 = w^4 (19 w - 3k^2) \tag{5}$$

The expression $(19 w - 3k^2)$ is a perfect square when

$$w = k^2 + 8kn + 19n^2 \tag{6}$$

and from (5) & (4),one has

$$\begin{aligned} u &= (k^2 + 8kn + 19n^2)^2 (19n + 4k) \\ v &= k\,(k^2 + 8kn + 19n^2)^2 \end{aligned} \tag{7}$$

Substituting (7) in (2) ,we get

$$\begin{aligned} x &= (k^2 + 8kn + 19n^2)^2 (19n + 5k) \\ y &= (k^2 + 8kn + 19n^2)^2 (19n + 3k) \\ z &= 2(k^2 + 8kn + 19n^2)^2 (19n + 4k) \end{aligned} \tag{8}$$

Thus, (6) & (8) satisfy (1).

Process 2

The choice

$$u = k w^2 \tag{9}$$

in (3) gives

$$3v^2 = w^4 (19w - k^2) \tag{10}$$

The expression $(19w - k^2)$ is a perfect square multiple of three when

$$w = 4k^2 + 30nk + 57n^2 \tag{11}$$

and from (10) & (9),one has

$$\begin{aligned} v &= (4k^2 + 30nk + 57n^2)^2 (19n + 5k) \\ u &= k(4k^2 + 30nk + 57n^2)^2 \end{aligned} \tag{12}$$

Substituting (12) in (2) ,we get

$$\begin{aligned} x &= (4k^2 + 30nk + 57n^2)^2 (19n + 6k) \\ y &= -(4k^2 + 30nk + 57n^2)^2 (19n + 4k) \\ z &= 2k(4k^2 + 30nk + 57n^2)^2 \end{aligned} \tag{13}$$

Thus, (11) & (13) satisfy (1).

Process 3

Let

$$w = a^2 + 3b^2 \tag{14}$$

Consider the integer 19 on the R.H.S. of (3) as

$$19 = (4 + i\sqrt{3})(4 - i\sqrt{3}) \tag{15}$$

Substitute (14) & (15) in (3) and employ the method of factorization. On equating the positive factors on both sides, one obtains

$$\begin{aligned} u + i\sqrt{3}\, v &= (4 + i\sqrt{3})(a + i\sqrt{3}\, b)^5 \\ &= (4 + i\sqrt{3})[f(a,b) + i\sqrt{3}\, g(a,b)] \end{aligned} \tag{16}$$

where

$$f(a,b) = a^5 - 30a^3b^2 + 45ab^4$$

$$g(a,b) = 5a^4b - 30a^2b^3 + 9b^5$$

On equating the coefficients of corresponding terms in (16), one obtains

$$u = 4f(a,b) - 3g(a,b)$$

$$v = f(a,b) + 4g(a,b)$$

In view of (2), we have

$$\begin{aligned} x &= 5f(a,b) + g(a,b) \\ y &= 3f(a,b) - 7g(a,b) \\ z &= 8f(a,b) - 6g(a,b) \end{aligned} \tag{17}$$

Thus, (14) & (17) satisfy (1).

Process 4

Write (3) as

$$u^2 + 3v^2 = 19w^5 * 1 \quad (18)$$

Consider the integer 1 on the R.H.S. of (18) as

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \quad (19)$$

Assume

$$w = 4(a^2 + 3b^2) \quad (20)$$

Substitute (15), (19) & (20) in (18) and employ the method of factorization. On equating the positive factors on both sides, one obtains

$$\begin{aligned} u + i\sqrt{3}\,v &= (4 + i\sqrt{3}) * 2^5 [f(a,b) + i\sqrt{3}\,g(a,b)] * \frac{(1 + i\sqrt{3})}{2} \\ &= 2^4 (1 + i5\sqrt{3})[f(a,b) + i\sqrt{3}\,g(a,b)] \end{aligned} \quad (21)$$

On equating the coefficients of corresponding terms in (21), one obtains

$$u = 2^4[f(a,b) - 15g(a,b)]$$

$$v = 2^4[5f(a,b) + g(a,b)$$

In view of (2), we have

$$\begin{aligned} x &= 2^4[6f(a,b) - 14g(a,b)] \\ y &= 2^4[-4f(a,b) - 16g(a,b) \\ z &= -f(a,b) - 3g(a,b) \end{aligned} \quad (22)$$

Thus, (20) & (22) satisfy (1).

Note 1

In addition to (19), the integer 1 has the following representations:

$$1 = \frac{(r^2 - 3s^2 + i\sqrt{3}(2rs))(r^2 - 3s^2 - i\sqrt{3}(2rs))}{(r^2 + 3s^2)^2}$$

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}(2rs))(3r^2 - s^2 - i\sqrt{3}(2rs))}{(3r^2 + s^2)^2}$$

$$1 = \frac{(6s^2 - 6s + 1 + i\sqrt{3}(2s - 1))(6s^2 - 6s + 1 - i\sqrt{3}(2s - 1))}{(6s^2 - 6s + 2)^2}$$

$$1 = \frac{(2s^2 - 2s - 1 + i\sqrt{3}(2s - 1))(2s^2 - 2s - 1 - i\sqrt{3}(2s - 1))}{(2s^2 - 2s + 2)^2}$$

Repeating the above process, one obtains four more sets of integer solutions to (1).

Conclusion

In this paper, an attempt has been made to obtain many integer solutions to non-homogeneous Diophantine equation of degree six with four unknowns. Substitution strategy and factorization technique are utilized for finding the required integer solutions to the sixth degree equation with four unknowns. In this analysis, the given equation is reduced to lower degree equation for which the integer solutions can be found elegantly.

References

[1] S.Vidhyalakshmi, J.Shanthi and M.A.Gopalan, Observation on the paper entitled Integral solution of the Homogeneous ternary cubic equation $x^3 + y^3 = 52(x + y)z^2$, International journal of Multidisciplinary research Volume 8, Issue 2, Page No.266-273 , February 2022.

[2] J.Shanthi , M.A.Gopalan, "On Non-Homogeneous cubic Equation with Four Unknowns $x^2 + y^2 + 4(35z^2 - 4 - 35w^2) = 6xyz$", Bio science Bio Technology Research Communication, Volume 14, Issue 05, 126- 129, March2021.

[3] J.Shanthi , M.A.Gopalan, "A Search on Non-Distinct Integer Solutions to Cubic Diophantine Equation with four unknowns $x^2 - xy + y^2 + 4w^2 = 8z^3$, International Research Journal of Education and Technology , Volume 2 Issue 01, Page No:27-32, May 2021.

[4] J.Shanthi , M.A.Gopalan, "On the non-homogeneous cubic Diophantine equation with four unknowns $x^2 + y^2 + 4((2k^2 - 2k)^2 z^2 - 4 - w^2) = (2k^2 - 2k + 1)xyz$ ", International journal of Mathematics and Computing Techiques, Volume 4, Issue 3, Page No:01-05, June 2021.

[5] J.Shanthi , S.Vidhyalakshmi , M.A.Gopalan , On Finding Integer Solutions to Sextic Equation with Three Unknowns $x^2 + y^2 = 8z^6$,The Ciencia and Engenharia-Science and Engineering Journal ,11(1) ,350-355 ,2023.

[6] M.A.Gopalan, Sharadha Kumar, "On the non-homogeneous ternary cubic equation $3(x^2 + y^2) - 5xy + x + y + 1 = 111z^3$", International Journal of Engineering and Techniques, 4(5), 105-107, Sep-Oct (2018).

[7] N.Thiruniraiselvi, M.A.Gopalan, Observations on $x^2 = y^{2a+1} + z^{2b+1}$, Global Journal of Engineering Science and Researches, Vol.5(10),Pp.162-168, 2018.

[8] Sharadha Kumar, M.A.Gopalan, "On the Cubic Equation $x^3 + y^3 + 6(x + y)z^2 = 4w^3$", JETIR, 6(1), 658-660, January 2019.

[9] N.Thiruniraiselvi, M.A.Gopalan, The Non-Homogeneous Quintic Equation with Six Unknowns $x^4 - y^4 = 109(z + w)P^3Q$, International Journal for Research in Applied Science and Engineering Technology, Vol.7(VI), June 2019.

[10] N.Thiruniraiselvi, M.A.Gopalan, The Non-Homogeneous Quintic Equation with Six Unknowns $(x^4 - y^4) = 109(z + w)P^3Q$, International Journal for Research in Applied Science & Engineering Technology ,7(VI), 2527-2530, 2019.

[11] A.Vijayasankar, Sharadha Kumar, M.A.Gopalan. On The Non-Homogeneous Quintic Equation with Five Unknowns $3(x+y)(x^3-y^3)=7(z^2-w^2)p^3$, 10(8), 44-49, 2020.

[12] N.Thiruniraiselvi, M.A.Gopalan, On the equations $y^2=11x^2+22$, Academic Journal of Applied Mathematical Sciences AJAMS, Vol 6(7), Pp.85-92, July 2020.

[13] N.Thiruniraiselvi, M.A.Gopalan, Observations on the Sextic Equation with three unknowns $3(X^2+Y^2)-2XY=972z^6$, International Journal of Mathematics, Statistics And Operations Research , Volume 1(2), Pp. 93-98 , 2021.

[14] N.Thiruniraiselvi, M.A.Gopalan, Observations on the Sextic Equation with three unknowns $3(x^2+y^2)-2xy=972z^6$, International Journal Of Mathematics, Statistics And Operations Research ,Volume 1; Number 2;,Pp. 93-98 ,2021.

[15] A.Vijayasankar, Sharadha Kumar, M.A.Gopalan, On Non-Homogeneous Quinary Quintic Equation $(x^4-y^4)=125(z^2-w^2)p^3$, South East Asian J. of Mathematics and Mathematical Sciences,18(1), 27-34, 2022.

[16] N.Thiruniraiselvi, M.A.Gopalan, Delineation of Integer Solutions to Homogeneous Bi-quadratic Equation with Four Unknowns $x^4+y^4+z^4=18w^4$, Journal of Liaoning Technical University, Vol 17(10), 2023.

[17] N.Thiruniraiselvi, M.A.Gopalan, Techniques to solve Diophantine Equation of Degree Ten with Six Unknowns $x^6-y^6-3456z^3=800(p^2-q^2)R^8$, IJO International Journal of Mathematics, Vol 7(5), Pg 1-7,2024.

[18] N.Thiruniraiselvi, M.A.Gopalan, Non-homogeneous Binary Cubic Equation $a(x-y)^3=8xy, a>0$, Bulletin of Pure and Applied Sciences Section - E - Mathematics & Statistics, 43E(1), 37–42 (2024).

[19] N.Thiruniraiselvi, Sharadha Kumar, M.A.Gopalan, Homogeneous Quinary Cubic Equation $(x^3-y^3)=(z^3-w^3)+90t^3$, Annals of Communications in Mathematics, Vol 7(2), Pg: 95-99, *2024.*

[20] N.Thiruniraiselvi, M.A.Gopalan, New Class of Integer Solutions to Ternary Quadratic Equation $x^2+y^2=(a^2+b^2)z^2$, Arhimede Mathematical Journal,Vol.11(1), 21-25, 2024.

[21] N.Thiruniraiselvi, M.A.Gopalan, Technique on Solving a Binary Quadratic Diophantine Equation $3x^2+5y^2=17^{(2k+1)}$, Indiana Journal of Multidisciplinary Research, 4(4): 1-3, 2024.

[22] N.Thiruniraiselvi, M.A.Gopalan,Technique on Solving a Binary Quadratic Diophantine Equation Involving Odd Powers of 23, International Journal of Research in Academic World, Vol 3(7), 196-198, (July 2024).

[23] N.Thiruniraiselvi, M.A.Gopalan, A Modish Glance of Integer Solutions to Non-homogeneous Cubic Diophantine Equation with Three Unknowns **$7(x^2-xy+y^2)=12z^3$**, International Journal of Current Science Research and Review, Vol 07(08), August 2024.

[24] N.Thiruniraiselvi, Sharadha Kumar, M.A.Gopalan, Second Degree Quinary Equation $xy+XY=(k^2-2k+1)w^2$,"Advances in Nonlinear Variational Inequalities", 27(2), 18-22, 2024.

[25] N.Thiruniraiselvi, M.A.Gopalan, A Search on Integer Solutions to Ternary Non-homogeneous Nonic Diophantine Equation, "Advances in Nonlinear Variational Inequalities", 27(2), 13-17, 2024.

[26] N.Thiruniraiselvi, M.A.Gopalan, On Finding Integer Solutions To Homogeneous Ternary Quadratic Diophantine Equation $x^2+(2k+1)y^2=(k+1)^2z^2$, International Journal of Recent Scientific Research Vol. 15, Issue, 09, pp.5003-5005, September, 2024.

[27] N.Thiruniraiselvi, M.A.Gopalan, A Search on Integer Solutions to Ternary Non-homogeneous Nonic Diophantine Equation $\alpha(x^2+y^2)-(2\alpha-1)xy=4\alpha z^9$, "Advances in Nonlinear Variational Inequalities", 27(2), 13-17, 2024.

[28] N.Thiruniraiselvi, M.A.Gopalan, Techniques to solve Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=7(z-w)^2(z+w)$, International Journal of Research – GRANTHAALAYAH, 12(10), 62–69, October 2024.

[29] R.Sathiyapriya, N.Thiruniraiselvi, M.A.Gopalan, A Glimpse on Homogeneous Ternary Quadratic $3x^2+2y^2=275z^2$, Journal of Computational Analysis and Applications, 33(7),693-696, 2024.

[30] R.Sathiyapriya, N.Thiruniraiselvi, M.A.Gopalan, Lattice points on the Cone $y^2+5x^2=86z^2$, Nanotechnology Perceptions, 20(7), 890–898, 2024.

[31] R.Sathiyapriya, N.Thiruniraiselvi, M.A.Gopalan, A Modish Glance of Integer Solutions to Nonhomogeneous Cubic Diophantine Equation with Three Unknowns $5(x^2+y^2)=13z^3$, *Nanotechnology Perceptions,* 21(1), (2025), 275-280, 2025.

[32] M.A.Gopalan, Sharadha Kumar, On Finding Integer Solutions on Binary Heptic Equation $x^2-xy^3=2y^7$,IRJEdT,8(7),782-787,2025.

[33] M.A.Gopalan,Sharadha Kumar, Techniques on solving Binary Heptic Equation $x^2-xy^3=4y^6+2y^7$,IJPREMS, 5(8), 91-95, 2025.

[34] N. Thiruniraiselvi, S. Devibala, J. Shanthi,Patterns of Integer Solutions to Non-Homogeneous Ternary Sextic Diophantine Equation, Boletim da Sociedade Paranaense de Matemática, Mathematics and Computing - Innovations and Applications, 2025 (43) 3, 1-7

[35] J.Shanthi, S.Devibala ,N.Thiruniraiselvi,M.A.Gopalan,Sharadhakumar,On solving non-homogeneous ternary higher degree diophantine equation, Boletim da Sociedade Paranaense de Matemática, Advances in Nonlinear Analysis and Applications, Vol 43, No 2 (2025).

[36] S.Devibala,J.Shanthi,N.Thiruniraiselvi,M.A.Gopalan,Sharadhakumar, A Trek on the Nonic Surface, Boletim da Sociedade Paranaense de Matemática, Advances in Nonlinear Analysis and Applications, (43) 2, Pg.1-6, 2025.

[37] N.Thiruniraiselvi, M.A.Gopalan, On Finding Integer Solutions to Homogeneous Ternary Quadratic Diophantine Equation $x^2+(2k+1)y^2=(k+1)^2z^2$, Rattanakosin Journal of Science and Technology: RJST, Volume 7Issue 3: 252-258, 2025.

[38] S.Devibala, N.Thiruniraiselvi, M.A.Gopalan, A Collection of Integer Solutions to Non-homogeneous Ternary Nonic Diophantine Equation $3(x^2 + y^2)-5xy=(k^2 + 11)z^9$, Gongcheng Kexue Xuebao, Volume 11(03), 104-110,2026.

[39] J.Shanthi, N.Thiruniraiselvi, M.A.Gopalan, Patterns of Integer Solutions to Non-homogeneous Ternary Heptic Diophantine Equation $x^2 + y^2 = (a^2 + b^2) z^7$, Gongcheng Kexue Xuebao, Volume 11(03),41-47, 2026.

[40] N.Thiruniraiselvi, J.Shanthi, M.A.Gopalan, A Glimpse on the Non-homogeneous Quaternary Octic Surface $x^3 + y^3 =2(a^2 + 3b^2) zw^7$, Gongcheng Kexue Xuebao, Volume 11(03), 88-95, 2026.

[41] N.Thiruniraiselvi, J.Shanthi, M.A.Gopalan, A Trek on the Non-homogeneous Quaternary Octic Surface $x^3 + y^3 =8zw^7$, Gongcheng Kexue Xuebao, Volume 11(03), 111-117, 2026.