Rotational Surfaces in the Real Special Linear Group $\mathbf{SL}(2,\mathbf{R})$ satisfying $\triangle r_i = \lambda_i r_i$

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Abstract

We completely classify all rotational surfaces in the Real Special Linear Group $\mathbf{SL}(2,\mathbf{R})$ whose coordinate functions are eigenfunctions of the Beltrami-Laplace operator associated with the left-invariant metric, i.e., satisfying $\triangle r_i = \lambda_i r_i$ for $\lambda_i \in \mathbb{R}$. Using the Iwasawa decomposition (x, y, θ) of $\mathbf{SL}(2,\mathbf{R})$ and the explicit form of the left-invariant metric, we derive a system of ordinary differential equations governing such surfaces. By analyzing this system according to the values of the eigenvalues λ_1, λ_2, and λ_3, we identify several distinct families of rotational surfaces. These include surfaces with constant y, surfaces whose generating curve is linear ($x = ay$), and surfaces whose generating curve satisfies quadratic relations of the form $\lambda_1 x^2 + (\lambda_2 - 4)y^2 = C_0$ or $4y^2 - \lambda_1 x^2 = C$, together with a first-order differential equation involving the slope $p = dx/dy$. The classification reveals explicit relations among the eigenvalues and highlights the affine and quadratic nature of the generating curves. A detailed example is provided to illustrate the theoretical results. This work extends classical results of Takahashi and Garay on finite-type surfaces to the non-trivial setting of the Lie group $\mathbf{SL}(2,\mathbf{R})$ equipped with its canonical left-invariant metric.

Keywords : Real Special Linear Group, Rotational surfaces, Beltrami-Laplace Operator, left-invariant metric, finite-type surfaces.
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1 Introduction

Many works were done to characterize the classification of submanifolds in terms of finite type. Important results about 2-type spherical closed submanifolds (where spherical means into a sphere) have been obtained in [3]. A well known result due to Takahashi [7], states that the minimal surfaces and the spheres are the only surfaces in $\mathbf{R}^3$ satisfying the condition

$$\triangle r = \lambda r, \quad \lambda \in \mathbf{R}.$$

where $\triangle$ is the Laplace operator associated with the induced metric.

On the other hand Garay [4] determined the complete surfaces of revolution in $\mathbf{R}^3$ whose component functions are eigenfunctions of their Laplace operator, i.e.

$$\triangle r_i = \lambda_i r_i, \quad \lambda_i \in \mathbf{R}. \tag{1.1}$$

Later the same author in [5] studied the hypersurfaces in $\mathbf{R}^{n+1}$ verifying

$$\triangle r = Ar, \quad A \in \mathbf{R}^{n+1 \times n+1}.$$

In [6] Kaimakamis and Papantouniou studied surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying the condition

$$\triangle^{II} r = Ar$$

where $\triangle^{II}$ is the Laplace operator with respect to the second fundamental form and A is a real 3×3 matrix.

In this paper. The present work aims to extend these classical studies to the setting of the non-abelian Lie group $\mathbf{SL}(2, \mathbf{R})$. Specifically, we classify all rotational surfaces in the Real Special Linear Group $\mathbf{SL}(2, \mathbf{R})$ whose coordinate functions are eigenfunctions of the Beltrami-Laplace operator associated with the left-invariant metric. This means we seek surfaces satisfying 1.1, where $\triangle$ is the Laplace-Beltrami operator of the induced metric. The choice of $\mathbf{SL}(2, \mathbf{R})$ is motivated by its fundamental role in geometry and physics, particularly its connection to hyperbolic geometry and its structure as a non-compact semi-simple Lie group. The classification reveals how the eigenvalues λ_i determine the nature of the generating curves and provides explicit relations among them.

The paper is organized as follows. In Section 2, we recall the necessary preliminaries on $\mathbf{SL}(2, \mathbf{R})$, including its Iwasawa decomposition (x, y, θ) and the explicit form of its canonical left-invariant metric. This coordinate system provides a convenient parametrization for studying rotational surfaces. Section 3 constitutes the core of the paper. We first derive the expressions for the induced metric components and the Laplace-Beltrami operator acting on the coordinate functions. This leads to a system of ordinary differential equations (3.8) that characterizes rotational surfaces satisfying $\triangle r_i = \lambda_i r_i$. We then analyze this system exhaustively according to the values of the eigenvalues λ_1, λ_2, and λ_3. This analysis yields several distinct families of surfaces, distinguished by the nature of their generating curves: linear curves ($x = ay$), curves satisfying quadratic relations of the form $\lambda_1 x^2 + (\lambda_2 - 4)y^2 = C_0$ or $4y^2 - \lambda_1 x^2 = C$, and curves defined by a first-order differential equation involving the slope $p = dx/dy$. The main results are summarized in Theorem 3.1. In Section 4, we present a detailed example illustrating the theoretical results, namely a rotational surface with linear generating curve corresponding to $\lambda_3 = 0$ and $a = 1$, yielding $\lambda_1 = \lambda_2 = 2$. All computations are verified explicitly, confirming consistency with the classification. Section 5 provides concluding remarks and discusses potential directions for future research.

2 Preliminaries

Let G denote the 2×2 real special linear group, which is a non-compact connected Lie group of dimension 3, i.e $G = \mathbf{SL}(2,\mathbf{R}) = \{g \in \mathbf{GL}(2,\mathbf{R});\ \ det\ g = 1\}$.

It is well-known that any element g of G can be written uniquely in the Iwasawa decomposition as

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad x \in \mathbf{R},\ y \in \mathbf{R}_*^+,\ \theta \in \mathbf{S}^1. \tag{2.1}$$

Thus G is homeomorphic to $\mathbf{R} \times \mathbf{R}_*^+ \times \mathbf{S}^1$ and we consider (x, y, θ) as a local coordinate system of G.

Let $\mathfrak{g}$ denote the Lie algebra of G, i.e, $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{R}) = \{X \in \mathfrak{gl}(2,\mathbf{R});\ \ tr\ X = 0\}$.
We define an inner product $<,>$ on $\mathfrak{g}$ by $2 < X, Y >= tr({}^tXY)$ where ${}^tX$ denotes the transposed matrix of X, in other words, take the following e_1, e_2, e_3 to be an orthonormal basis:

$$e_1 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We identify $\mathfrak{g}$ with the tangent space T_eG at the identity element $e \in G$ and define a metric on G by the left translation in the usual manner. We denote this left invariant metric by ds_G^2. Using the coordinate (x, y, θ) we endow $G = \mathbf{SL}(2,\mathbf{R})$ with its canonical left-invariant Riemannian metric

$$ds_G^2 = \left(\frac{dx}{2y}\right)^2 + \left(\frac{dy}{2y}\right)^2 + \left(\frac{dx}{2y} + d\theta\right)^2. \tag{2.2}$$

The metric ds_G^2 is represented by the tensor matrix ϱ given by

$$\varrho = \begin{pmatrix} \frac{1}{2y^2} & 0 & \frac{1}{2y} \\ 0 & \frac{1}{4y^2} & 0 \\ \frac{1}{2y} & 0 & 1 \end{pmatrix}$$

The metric on $G/\mathbf{SO}(2)$ associated with ds_G^2 (cf. [2]) is exactly the Poincaré metric of constant Gauss curvature -4.

Let $\{\omega^1,\ \omega^2,\ \omega^3\}$ be an orthonormal coframe field defined by

$$\omega^1 = \frac{1}{2y}dx, \quad \omega^2 = \frac{1}{2y}dy, \quad \omega^3 = \frac{1}{2y}dx + d\theta, \tag{2.3}$$

and $\{e_1, \ e_2, \ e_3\}$ be the orthonormal frame field dual to $\{\omega^1, \ \omega^2, \ \omega^3\}$ given by

$$e_1 = 2y\frac{\partial}{\partial x} - \frac{\partial}{\partial \theta}, \quad e_2 = 2y\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial \theta}. \tag{2.4}$$

Note that $\{\omega^j\}$ and $\{e_i\}$ are globally defined on G.
By differentiating the equations (2.3) we have $d\omega^1 = d\omega^3 = 2\omega^1 \wedge \omega^2 = d\omega^1, \ \ d\omega^2 = 0$.

Hence we have the structure equations

$$d\begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\end{bmatrix} = -\begin{bmatrix}0 & -2\omega^1 - \omega^3 & -\omega^2\\ 2\omega^1 + \omega^3 & 0 & \omega^1\\ \omega^2 & -\omega^1 & 0\end{bmatrix} \wedge \begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\end{bmatrix} \tag{2.5}$$

So

$$\omega = \begin{bmatrix}0 & -2\omega^1 - \omega^3 & -\omega^2\\ 2\omega^1 + \omega^3 & 0 & \omega^1\\ \omega^2 & -\omega^1 & 0\end{bmatrix} \tag{2.6}$$

is the connection matrix for the Levi-Civita connection of ds^2_G.

3 Rotational surfaces

Definition 3.1. *An immersed surface $r : M \to G$ is said to be* ***rotational*** *if it is invariant under the left translation of the subgroup $\mathbf{SO}(2)$ of G, i.e., $r(M)\mathbf{SO}(2) \subset r(M)$.*

Obviously a **rotational surface** has the following parametrisation :

$$r : (t, \theta) \mapsto \begin{pmatrix}1 & x(t)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y(t)} & 0\\ 0 & \frac{1}{\sqrt{y(t)}}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

where $(t, \theta) \in I \times \mathbf{S}^1$. ($I$ denotes an interval of $\mathbf{R}$).

The condition $x'(t)^2 + y'(t)^2 \neq 0$ is necessary for r to be an immersion. We call the curve $(x(t), y(t))$ on $S_0 = \{\theta = 0\}$ the generating curve of r.

we denote by $E, F, G; L, M, N$ the coefficients of the first and second fundamental form, respectively, of the rotational surface.

If $\phi : M \to \mathbf{R}$, $(t, \theta) \mapsto \phi(t, \theta)$ is a smooth function and $\triangle$ the Beltrami-Laplace operator with respect to the first fundamental form of M, then from [8] we have

$$\triangle\phi = -\frac{1}{\sqrt{|EG-F^2|}}\left[\left(\frac{G\phi_t - F\phi_\theta}{\sqrt{|EG-F^2|}}\right)_t - \left(\frac{F\phi_t - E\phi_\theta}{\sqrt{|EG-F^2|}}\right)_\theta\right] \tag{3.1}$$

The mean curvature $\mathbf{H}$ and the Gaussian curvature $\mathbf{K}_G$ are given by

$$\mathbf{H} = \frac{GL + EN - 2FM}{2(EG - F^2)} \quad and \quad \mathbf{K}_G = \frac{LN - M^2}{EG - F^2} \tag{3.2}$$

We explore now the classification of the rotational surfaces M satisfying the relation (1.1). The surface M is parametrized by

$$r : (t,\theta) \mapsto \begin{pmatrix} 1 & x(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y(t)} & 0 \\ 0 & \frac{1}{\sqrt{y(t)}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

So in the local coordinates (x, y, θ) we have

$$r(t,\theta) = (x(t), y(t), \theta) \tag{3.3}$$

$x(t)$ and $y(t)$ being smooth functions of the variable $t \in I$. For the parametrization to be an immersion, the condition $x'(t)^2 + y'(t)^2 \neq 0$ is necessary, where $(')$ means differentiation with respect to t.

So we have natural frame $\{r_t, r_\theta\}$ given by

$$r_t = (x'(t), y'(t), 0) \quad and \quad r_\theta = (0, 0, 1). \tag{3.4}$$

where $x'(t)$ and $y'(t)$ mean the differentiation of $x(t)$ and $y(t)$ respectively.

The induced metric on M is obtained by

$$E = \varrho(r_t, r_t) = \frac{2x'(t)^2 + y'(t)^2}{4y(t)^2}, \qquad F = \varrho(r_t, r_\theta) = \frac{x'(t)}{2y(t)}, \qquad G = \varrho(r_\theta, r_\theta) = 1. \tag{3.5}$$

If this rotational surface is constructed with component functions which are eigenfunctions of its Laplacian, then we shall have

$$\begin{cases} \triangle x(t) = \lambda_1 x(t) \\ \triangle y(t) = \lambda_2 y(t) \\ \triangle\theta = \lambda_3\theta \end{cases} \tag{3.6}$$

where λ_1, λ_2, $\lambda_3 \in Spec(M)$. So we get

$$\begin{cases} \Delta x &= \dfrac{4y\left[x'y'W^2 + yy'K\right]}{W^4}, \\ \Delta y &= \dfrac{4y\left[y'^2W^2 - x'yK\right]}{W^4}, \\ \Delta\theta &= \dfrac{2yy'K}{W^4}. \end{cases} \tag{3.7}$$

where $W=\sqrt{x'^2+y'^2}$ and $K=x''y'-x'y''$.
The relation (1.1) by means of (3.3) , (3.6) and (3.7) gives rise to the following system

$$\begin{cases} 4y\left(x'y'W^2+yy'K\right) &= \lambda_1 x W^4, & (a)\\ 4y\left(y'^2W^2-x'yK\right) &= \lambda_2 W^4, & (b)\\ 2yy'K &= \lambda_3\theta W^4. & (c)\end{cases} \tag{3.8}$$

Therefore, the problem of classifying the rotational surfaces M satisfying (1.1) is reduced to the integration of the system (3.8) of ordinary differential equations. Next we study it according to the values of the constants λ_1, λ_2, λ_3.

3.1 Case I: $\lambda_3 = 0$.

Then the equation (3.8)(c) gives rise to $y'K=0$ where implies $y'=0$ or $K=0$.

-If $y'=0$ then $y(t)=y_0$ constant, $x(t)$ is arbitrary. We get also, by the equations (3.8)(a) and (3.8)(b), $\lambda_1=\lambda_2=0$.

- If $K=x''y'-x'y''=0$ implies $\dfrac{y''}{y'}=\dfrac{x''}{x'}$ which implies $x(t)=ay(t)+b$. Subtitution in (3.8)(a) and (3.8)(b) gives $\lambda_1=\lambda_2=\dfrac{4}{1+a^2}$ and $b=0$.

3.2 Case II: $\lambda_3 \neq 0$.

3.2.1 If $\lambda_1=\lambda_2=0$.

The system (3.8) is equivalently reduced to the system

$$\begin{cases} x'y'W^2+yy'K &= 0, & (i)\\ y'^2W^2-x'y &= 0, & (ii)\end{cases} \tag{3.9}$$

we multiply equation (3.9)(i) by x' and (3.9)(ii) by y' , and by addition we get $y'W^4=0$, this means that $y=y_0$ constant.

3.2.2 If $\lambda_1 \neq 0, \lambda_2 = 0$.

The system (3.8) is equivalently reduced to the system

$$\begin{cases} 4y\left(x'y'W^2+yy'K\right) &= \lambda_1 x W^4, & (i)\\ y'^2W^2-x'yK &= 0, & (ii)\\ 2yy'K &= \lambda_3\theta W^4. & (iii)\end{cases} \tag{3.10}$$

we multiply equation (3.10)(i) by x' and (3.10)(ii) by $4yy'$, and by addition we get

$$4yy'=\lambda_1 x x',$$

this equation has the following solution

$$4y^2 - \lambda_1 x^2 = C.$$

with a certain constant $C \in \mathbf{R}$.

From the equation (3.8)(c) we have $K = \dfrac{\lambda_3\theta(x'^2 + y'^2)^4}{2yy'}$.

Define $p = \dfrac{dx}{dy}$ then $x' = py'$ implies $x'' = p'y'^2 + py''$.
Thus

$$K = y'^3 \frac{dp}{dy} \tag{3.11}$$

By using (3.11) and subtitution in (3.8)(c) we obtain

$$\frac{dp}{dy} = \frac{\lambda_3\theta(1+p^2)^2}{2y}$$

This gives

$$\frac{dp}{(1+p^2)^2} = \frac{\lambda_3\theta}{2y}dy$$

Integration gives

$$\frac{p}{1+p^2} + \arctan p = \lambda_3\theta \ln y + C$$

3.2.3 If $\lambda_1 = 0, \lambda_2 \neq 0$.

The system (3.8) is equivalently reduced to the system

$$\begin{cases} x'W^2 + yK &= 0, \qquad (i) \\ y'^2W^2 - x'yK &= \lambda_2 W^4, \qquad (ii) \\ 2yy'K &= \lambda_3\theta W^4. \qquad (iii) \end{cases} \tag{3.12}$$

Let's multiply the equation (3.12)(i) by x' and subtitution in (3.12)(ii) we get

$$4\left(y'^2W^2 + x'^2W^2\right) = \lambda_2 W^4,$$

which implies $\lambda_2 = 4$. The equation (3.12)(iii) gives the same solution as previous, i.e

$$\frac{p}{1+p^2} + \arctan p = \lambda_3\theta \ln y + C$$

3.2.4 If $\lambda_1 \neq 0, \lambda_2 \neq 0$.

Let us now consider the system composed of equations (3.8)(a) and (3.8)(b) where the unknowns are W^2 and K. Therefore, this system is linear with respect to W^2 and K. The determinant of this system is equal to $-16y'y^2W^2$.

So

$$W^2 = \frac{1}{4yy'}\left(\lambda_1 xx'W^2 + \lambda_2 yy'W^2\right)$$

implies

$$\lambda_1 xx' + (\lambda_2 - 4)yy' = 0$$

Hence the solution is

$$\lambda_1 x^2 + (\lambda_2 - 4)y^2 = C_0.$$

with a certain constant $C_0 \in \mathbf{R}$.
So the solutions of the system (3.8) is

$$\begin{cases} \lambda_1 x^2 + (\lambda_2 - 4)y^2 = C_0 \\ \dfrac{p}{1+p^2} + \arctan p = \lambda_3 \theta \ln y + C \end{cases}, \tag{3.13}$$

Thus, we have just proved the following theorem

Theorem 3.2. *Let M be a rotational surface given by (3.3). Then $\triangle r_i = \lambda_i r_i$ if and only if the following statements hold true*

- *M is parametrized by $r(t,\theta) = (x(t), y_0, \theta)$, $y_0 > 0$, $x(t)$ arbitrary smooth function.*
- *M is parametrized by $r(t,\theta) = (ay(t), y(t), \theta)$, $a \in \mathbf{R}$, $y(t)$ arbitrary smooth function.*
- *The generating curve $(x(t), y(t))$ of the surface M is defined by the equation $4y(t)^2 - \lambda_1 x(t)^2 = C$ where $C \in \mathbf{R}$.*
- *The generating curve $(x(t), y(t))$ of the surface M is defined by the equation $\lambda_1 x(t)^2 + (\lambda_2 - 4)y(t)^2 = C_0$ and the solution of $\dfrac{p}{1+p^2} + \arctan p = \lambda_3 \theta \ln y + C$ where $C,\ C_0 \in \mathbf{R}$.*

4 Examples

Example: A Rotational Surface with $\lambda_3 = 0$ and Linear Generating Curve

We present a detailed example illustrating the classification results.

Context from the Paper

In **Case I** of Section 3.1, we consider $\lambda_3 = 0$ and $K = 0$, where

$$K = x''(t)y'(t) - x'(t)y''(t)$$

The condition $K = 0$ implies

$$\frac{x''}{x'} = \frac{y''}{y'} \quad \Rightarrow \quad x(t) = ay(t) + b, \quad a, b \in \mathbb{R}$$

Substituting into equations (3.8)(a) and (3.8)(b) yields

$$\lambda_1 = \lambda_2 = \frac{4}{1+a^2}, \quad b = 0$$

Thus, the generating curve is a line through the origin

$$x(t) = a\,y(t)$$

Choice of Explicit Functions

We choose

- $a = 1$ (slope 1)
- $y(t) = t$, with $t > 0$ (so that $y \in \mathbb{R}^+$ as required by the Iwasawa decomposition)

Then,

$$x(t) = t$$

So the generating curve is

$$(t, t), \quad t > 0$$

Parametrization of the Rotational Surface

From equation (3.3) in the paper, a rotational surface in $\mathbf{SL}(2, \mathbf{R})$ is parametrized as

$$r(t, \theta) = (x(t), y(t), \theta), \quad t \in I \subset \mathbb{R},\ \theta \in S^1$$

Substituting our functions

$$r(t, \theta) = (t, t, \theta), \quad t > 0,\ \theta \in [0, 2\pi)$$

This is a surface in $\mathbf{SL}(2, \mathbf{R})$ via the Iwasawa coordinates (x, y, θ).

First Fundamental Form (Induced Metric)

From equation (3.5).

$$E = \varrho(r_t, r_t) = \frac{2x'(t)^2 + y'(t)^2}{4y(t)^2}$$

$$F = \varrho(r_t, r_\theta) = \frac{x'(t)}{2y(t)}$$

$$G = \varrho(r_\theta, r_\theta) = 1$$

Compute derivatives

$$x'(t) = 1, \quad y'(t) = 1, \quad y(t) = t$$

Then,

$$E = \frac{2(1)^2 + (1)^2}{4t^2} = \frac{2+1}{4t^2} = \frac{3}{4t^2}$$

$$F = \frac{1}{2t}$$
$$G = 1$$

Thus, the induced metric on the surface is

$$ds^2 = E\,dt^2 + 2F\,dt\,d\theta + G\,d\theta^2 = \frac{3}{4t^2}dt^2 + \frac{1}{t}dt\,d\theta + d\theta^2$$

Verification of the Eigenvalue Condition

We use equation (3.7) from the paper to compute the Laplacian of the coordinate functions.

$$\triangle x = \frac{4y\,[x'y'W^2 + yy'K]}{W^4}$$
$$\triangle y = \frac{4y\,[y'^2W^2 - x'yK]}{W^4}$$
$$\triangle \theta = \frac{2yy'K}{W^4}$$

where $W^2 = x'^2 + y'^2$ and $K = x''y' - x'y''$.

For our example,

$$x' = 1, \quad y' = 1, \quad x'' = 0, \quad y'' = 0 \quad \Rightarrow \quad K = 0, \quad W^2 = 1^2 + 1^2 = 2, \quad y = t$$

Then,

$$\triangle x = \frac{4t\,[1 \cdot 1 \cdot 2 + t \cdot 1 \cdot 0]}{2^2} = \frac{4t \cdot 2}{4} = 2t$$
$$\triangle y = \frac{4t\,[1^2 \cdot 2 - 1 \cdot t \cdot 0]}{4} = \frac{4t \cdot 2}{4} = 2t$$
$$\triangle \theta = \frac{2t \cdot 1 \cdot 0}{4} = 0$$

Since $x(t) = t$ and $y(t) = t$, we have

$$\triangle x = 2x, \quad \triangle y = 2y, \quad \triangle \theta = 0$$

Therefore,

$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = 0$$

which matches the paper's formula $\lambda_1 = \lambda_2 = \frac{4}{1+a^2} = \frac{4}{1+1} = 2$.

Result

The surface

$$r(t, \theta) = (t, t, \theta), \quad t > 0,\ \theta \in S^1$$

is a rotational surface in $\mathbf{SL}(2, \mathbf{R})$ satisfying $\triangle r_i = \lambda_i r_i$ with $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 0$. This corresponds to **Case I** in the paper, with $\lambda_3 = 0$, $K = 0$, and the generating curve $x(t) = ay(t)$ where $a = 1$. The surface is explicitly given, and all computations are consistent with the paper's framework.

Conclusion

In this paper, we have presented a complete classification of rotational surfaces in the Real Special Linear Group $\mathbf{SL}(2,\mathbf{R})$ whose coordinate functions are eigenfunctions of the Beltrami-Laplace operator associated with the left-invariant metric, i.e., satisfying $\triangle r_i = \lambda_i r_i$ for $\lambda_i \in \mathbb{R}$.

Using the Iwasawa decomposition (x, y, θ) of $\mathbf{SL}(2,\mathbf{R})$ and the explicit expression of the left-invariant metric, we derived the system of ordinary differential equations (3.8) that governs such surfaces. By analyzing this system according to the values of the eigenvalues λ_1, λ_2, and λ_3, we identified several distinct families of rotational surfaces:

- **Case I ($\lambda_3 = 0$):** This case splits into two subfamilies. The first consists of surfaces with constant y ($y(t) = y_0$), where $x(t)$ is arbitrary and $\lambda_1 = \lambda_2 = 0$. The second consists of surfaces whose generating curve is linear ($x(t) = ay(t)$), with $\lambda_1 = \lambda_2 = \frac{4}{1+a^2}$.
- **Case II ($\lambda_3 \neq 0$):** This case further subdivides according to the values of λ_1 and λ_2. When $\lambda_1 = \lambda_2 = 0$, we obtain surfaces with constant y. When $\lambda_1 \neq 0$ and $\lambda_2 = 0$, the generating curve satisfies $4y^2 - \lambda_1 x^2 = C$, along with a differential relation involving $p = dx/dy$. When $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, the generating curve satisfies $\lambda_1 x^2 + (\lambda_2 - 4)y^2 = C_0$, together with the same differential relation for p.

The classification reveals that the generating curves of these rotational surfaces are remarkably simple: they are either straight lines, circles, hyperbolas, or more general curves satisfying a first-order differential equation. The eigenvalues λ_i are not arbitrary but satisfy specific relations depending on the geometry of the generating curve.

A detailed example was provided to illustrate the theoretical results, demonstrating a rotational surface with linear generating curve corresponding to $\lambda_3 = 0$, $a = 1$, yielding $\lambda_1 = \lambda_2 = 2$. The explicit computation of the induced metric and the Laplacian of the coordinate functions confirmed the satisfaction of the eigenvalue condition.

This work extends the classical results of Takahashi and Garay to the non-trivial setting of the Lie group $\mathbf{SL}(2,\mathbf{R})$ equipped with its canonical left-invariant metric. The results contribute to the broader program of classifying finite-type submanifolds in homogeneous spaces. Future research could explore analogous classifications for other Lie groups or for surfaces satisfying higher-order finite-type conditions.

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