A Sparkle of Integer Solutions to Non-homogeneous Ternary Cubic Diophantine Equation

$a\,(x^2+y^2)=b\,z^3,\ \text{g.c.d}(a,b)=1$

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Abstract

The main thrust of this paper is focused on obtaining varieties of integer solutions to the third degree Diophantine equation with three unknowns $a(x^2+y^2)=b\,z^3, \text{g.c.d}(a,b)=1$.Various choices of solutions in integers are obtained by reducing it to the equation which is solvable through employing suitable transformations and applying the factorization method. Also, the integer solutions to the negative pellian equation $y^2=2x^2-1$ are utilized to obtain many integer solutions to the considered cubic equation.

Keywords: ternary cubic equation ,non-homogeneous cubic equation, integer solutions, substitution technique , factorization method, negative pellian equation

Introduction

One of the interesting areas of Theory of Numbers is the study of diophantine equations which has fascinated and motivated both Amateurs and Mathematicians alike. It is well-known that diophantine equation is a polynomial equation in two or more unknowns requiring only integer solutions. Obviously, Diophantine equations are plenty. The theory of Diophantine equations is popular in recent years providing a fertile ground for both Professionals and Amateurs. In addition to known results, this abounds with unsolved problems. Although many of its results can be stated in simple and elegant terms, their proofs are sometimes long and complicated.. The successful completion of presenting solutions in integers satisfying the requirements set forth in the problem add to the improvement of number theory as they offer good applications in the field of Graph theory, Modular theory, Coding and Cryptography, Engineering, Music and so on. The theory of integers provides answers to real world problems.

Obviously, homogeneous or not-homogeneous equations have motivated various mathematicians. It is worth to observe that Cubic Diophantine equations fall in to the theory of Elliptic curves which are used in Cryptography. In particular, one may refer [1-15] for third degree equations with multiple variables.

The present paper aims in illustrating different choices of solutions in integers to an interesting ternary non- homogeneous cubic equation $a(x^2+y^2)=bz^3, g.c.d(a,b)=1$ through employing elementary algebraic methods.

Method of analysis

The non-homogeneous third degree Diophantine equation with three unknowns to be solved is

$$a\,(x^2+y^2)=bz^3 \qquad (1)$$

In the above equation (1) ,the values of a and b are such that g,c,d, of (a,b)=1.

Different approaches of obtaining distinct integer solutions to (1) are exhibited below.

Approach 1

The substitution

$$x=ky \qquad (2)$$

in (1) leads to

$$a(k^2+1)\,y^2=b\,z^3 \qquad (3)$$

whose solutions are

$$y=a*b^2*(k^2+1)t^{3s}, z=a*b*(k^2+1)t^{2s} \qquad (4)$$

Using (4) in (2), we get

$$x = a * b^2 * k(k^2 + 1)t^{3s} \quad (5)$$

Thus, (4) & (5) represent the integer solutions to (1).

Approach 2

Taking

$$x = a * b^2 * w, z = a * b * w \quad (6)$$

in (1), it leads to

$$y^2 = a^2 * b^4 * (w - 1)w^2 \quad (7)$$

The R.H.S. of (7) is a perfect square when

$$w = (s^2 + 1) \quad (8)$$

Employing (8) in (6) & (7), the respective solutions in integers to (1)

are given by

$$x = a * b^2 * (s^2 + 1), y = a * b^2 * s(s^2 + 1), z = a * b * (s^2 + 1)$$

Approach 3

Taking

$$x = a * b * k * w, z = a * w \quad (9)$$

in (1), it leads to

$$y^2 = a^2 * w^2 * (b * w - b^2 * k^2) \quad (10)$$

The R.H.S. of (10) is a perfect square when

$$w = b * k^2 * (s^2 + 1) \quad (11)$$

Employing (11) in (9) & (10),the respective solutions in integers to (1)

are given by

$$x = a * b^2 * k^3 * (s^2 + 1),$$
$$y = a * b^2 * k^3 * s * (s^2 + 1),$$
$$z = a * b * k^2 * (s^2 + 1).$$

Note 1

It is worth to mention that , in (10) , the expression $(b*w-b^2*k^2)$ is a perfect square for the values of w given by

$$w=w_n=b*n^2+2*n*b*k+2*b*k^2$$

and

$$b*w-b^2*k^2=(b*n+b*k)^2=(\alpha_n)^2 \text{ say}$$

Thus , from (9) and (10) ,we get

$$x_n=a*b*k*w_n=a*b*k*(b*n^2+2*n*b*k+2*b*k^2),$$
$$z_n=a*w_n=a*(b*n^2+2*n*b*k+2*b*k^2),$$
$$y_n=a*w_n*\alpha_n=a*(b*n^2+2*n*b*k+2*b*k^2)*(b*n+b*k).n=1,2,3,...$$

which gives sequence of integer solutions to (1).

Approach 4 (Special case)

Assume

$$z=A^2+B^2=(A+iB)(A-iB) \tag{12}$$

Choose the integers a and b in (1) such that one may write

$$a=p^2+q^2=(p+iq)(p-iq) \tag{13}$$

and

$$b=r^2+s^2=(r+is)(r-is) \tag{14}$$

Substitute (12), (13) and (14) in (1). Using factorization and equating the positive factors, we have

$$\begin{aligned}(p+iq)(x+iy)&=(r+is)(A+iB)^3\\&=(r+is)\,[f(A,B)+ig(A,B)]\end{aligned} \tag{15}$$

where

$$f(A,B)=A^3-3AB^2$$
$$g(A,B)=3A^2B-B^3$$

On comparing the real and imaginary parts in (15) ,we have

$$
\begin{aligned}
px - qy &= r\,f(A,B) - s\,g(A,B) \\
qx + py &= s\,f(A,B) + r\,g(A,B)
\end{aligned}
\tag{16}
$$

Solving the system of equations (16) for x and y, one has

$$
\begin{aligned}
(p^2+q^2)x &= p\,[r\,f(A,B) - s\,g(A,B)] + q[s\,f(A,B) + r\,g(A,B)] \\
(p^2+q^2)y &= p\,[s\,f(A,B) + r\,g(A,B)] - q[r\,f(A,B) - s\,g(A,B)]
\end{aligned}
\tag{17}
$$

As we require the solutions in integers, replacing A by $(p^2+q^2)u$ and B by $(p^2+q^2)v$ in (17) & (12), respective solutions in integers to (1) are given by

$$
\begin{aligned}
x &= (p^2+q^2)^2\{(pr+qs)f(u,v) + (qr-ps)g(u,v)\}, \\
y &= (p^2+q^2)^2\{(ps-qr)f(u,v) + (pr+qs)g(u,v)\}, \\
z &= (p^2+q^2)^2(u^2+v^2).
\end{aligned}
$$

Approach 5

Introduction of the transformations

$$x = 2*a*b^2*w^2, z = 2*a*b*w^2 \tag{18}$$

in (1) leads to

$$y^2 = 4*a^2*b^4*w^4*(2w^2-1) \tag{19}$$

Assume

$$\alpha^2 = 2w^2 - 1 \tag{20}$$

which is the well-known negative Pellian equation. Using the standard procedure, the general solution (α_{n+1}, w_{n+1}) to (20) is given by

$$
\begin{aligned}
w_{n+1} &= \frac{1}{4}[2f_n + \sqrt{2}\,g_n], \\
\alpha_{n+1} &= \frac{1}{2}[f_n + \sqrt{2}\,g_n]
\end{aligned}
\tag{21}
$$

Thus, from (18) & (19), the respective solutions in integers to (1) are

given by

$$x_{n+1} = 2*a*b^2*[\frac{2f_n+\sqrt{2}g_n}{4}]^2,$$
$$y_{n+1} = 2*a*b^2*[\frac{2f_n+\sqrt{2}g_n}{4}]^2*(\frac{f_n+\sqrt{2}g_n}{2}),$$
$$z_{n+1} = 2*a*b*[\frac{2f_n+\sqrt{2}g_n}{4}]^2$$

Conclusion:

The ternary non- homogeneous cubic represented by $a(x^2+y^2)=bz^3$, g.c.d $(a,b)=1$ is analysed for obtaining varies choices of solutions in integers. The readers of this paper may consider other forms of cubic equations with three or more variables for determining their solutions in integers.

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