Hamilton Semigraphs: Structural Properties and Theoretical Insights

Dr.Hanumesha A G, Associate professor, Department of Mathematics, SJC Institute of Technology, , Chickballapur-562101.

ABSTRACT

Hamiltonian cycles are a cornerstone of graph theory, closely linked to connectivity, optimization, and traversal problems [1], [2]. Extending Hamiltonian concepts to generalized structures is essential for accurately modeling complex networks that cannot be represented by simple graphs alone [18]. This paper introduces Hamilton Semigraphs and investigates their fundamental properties. Necessary and sufficient conditions for Hamiltonicity in semigraphs are established, along with structural characterizations that extend classical Hamiltonian criteria. All results are supported by rigorous theoretical proofs. The proposed framework broadens the scope of Hamiltonian theory and lays the groundwork for further applications in generalized network models.

KEYWORDS Hamilton semigraphs, Hamiltonian properties, semigraph theory, generalized graph structures, Hamiltonicity.

1. INTRODUCTION

Hamiltonian concepts occupy a central place in graph theory, bridging combinatorial design with both rigorous theoretical developments and a wide range of practical applications. A Hamiltonian cycle— characterised as a closed walk that traverses each vertex in a graph exactly once—has long been a subject of extensive study in classical graph theory. Landmark results, including Dirac's Theorem [1], Ore's Theorem [2], and the Bondy–Chvátal Closure Theorem [3], provide foundational insights into Hamiltonicity. These theorems not only deliver precise characterizations but also influence diverse areas such as algorithm design, combinatorial optimization, and computational complexity. Moreover, Hamiltonian structures have significant relevance in practice, with notable applications in network reliability, communication design, and routing problems in distributed environments [4], [5].

Despite these advances, many real-world networks—such as those arising in biological systems, social interactions, and communication infrastructures—exhibit relationships that extend beyond simple pairwise connections. Traditional graph-based models are often inadequate to represent such higher-order dependencies. This motivates the study of semigraphs, a generalized framework positioned between graphs and hypergraphs, where adjacency rules are relaxed to capture richer patterns of interaction [6]–[9].

In this work, we introduce and analyze the concept of Hamilton Semigraphs, which extend Hamiltonian cycles into the Semigraph paradigm. We establish sufficient and necessary requirements for Semigraph Hamiltonicity., establish structural results, and adapt key classical theorems to this generalized setting. In particular, we present a Bondy–Chvátal type closure theorem for Semigraphs, providing a direct connection between Hamiltonian graph theory and Semigraph models [10]–[12].

Beyond theoretical significance, Hamilton Semigraphs have the potential to support applications in areas such as robust communication networks, parallel computing, and the modeling of complex systems [13]–[15]. By embedding Hamiltonian principles into Semigraph theory, this study contributes to a broader and a more adaptable structure that combines classical results while opening pathways for further exploration in generalized network structures.

2. PRELIMINARIES

Definition 2.1

SEMIGRAPH

A finite set of edges (E) and a finite set of vertices (V) constitute a Semigraph S= (V,E), where each edge represents a non-empty finite subset of V. In contrast to traditional graphs, edges in a Semigraph may coincide with one another and incidences of Vertices are not limited to simply binary relationships.

Definition 2.2 (Hamiltonian Cycle in Semigraph).

In a Semigraph, a Hamiltonian cycle is a closed traversal that precisely traverses each vertex of V once before returning to its starting vertex.

Definition 2.3 (Hamilton Semigraph).

A Semigraph is termed a Hamilton Semigraph if it possesses at least one Hamiltonian cycle.

3. THEOREMS

Theorem 3.1 (Necessary Condition).

If a Semigraph S= (V,E) contains a Hamiltonian cycle, then every vertex $v \in V$ must have degree at least 2.

Proof.

Suppose a Hamiltonian cycle exists in S. By definition, the cycle must enter and exit each vertex exactly once. Thus, every vertex v must be incident with at least two edges: one for entry and one for exit. Therefore, $deg(v) \geq 2$ for all $v \in V$.

Theorem 3.2 (Sufficient Condition for 2-Regular Semigraphs).

If S= (V,E) is a connected Semigraph where When all of the vertices have degrees of exactly 2, S is a Hamilton Semigraph.

Proof.

If the degree of each vertex 2, then the Semigraph decomposes into one or more disjoint cycles. Since S is connected, there can be only one cycle covering all vertices. This cycle is Hamiltonian. Hence, S is a Hamilton Semigraph.

Lemma. Let S be a Semigraph of order n. Let $u,v \in V$ be distinct nonadjacent vertices with $deg_S(u)+deg_S(v) \geq n$. Let $S'=S+uv$ be the Semigraph obtained by adding an edge making u and v adjacent. Then S is Hamiltonian if and only if S′ is Hamiltonian.

Proof.

If S is Hamiltonian, any Hamiltonian cycle of S is also a Hamiltonian cycle of S′ (adding edges cannot destroy an existing cycle). Thus S′ is Hamiltonian.

Conversely, suppose S′ has a Hamiltonian cycle C. If C does not use the newly added edge uv, then We're done because C is already a Hamiltonian cycle of S. So assume C uses the edge uv. Remove the edge uv from C; what remains is a Hamiltonian path P in S whose endpoints are u and v. Write the vertices of P in order.

$P: u=v_0 , v_1 , v_2,\dots, v_{n-2} , v_{n-1}=v$

(Thus P visits every vertex exactly once.)

For each internal vertex v_i (where $1 \leq I \leq n-2$)

Define:

$A=\{ v_i: u \text{ is adjacent to } v_i \text{ in } S \}$, $B=\{ v_j: v \text{ is adjacent to } v_j \text{ in } S \}$.

Observe , $|A| \leq deg_S(u)$ and $|B| \leq deg_S(v)$, and $A,B \subseteq \{v_1,\dots,v_{n-2}\}$.

Suppose there exists an index i with $1\leq I \leq n-2$ such that $v_i \in A$ and $v_{i+1} \in B$. Then we can form a Hamiltonian cycle of S that avoids the added edge uv as follows:

$u\ (=v_0) - v_i - v_{i-1} - \dots - v_1 - v - v_{n-2} - v_{n-3} - \dots - v_{i+1} - u,$

Where each consecutive pair is adjacent in S because

u is adjacent to v_i (by $v_i \in A$),

v is adjacent to v_{i+1} (by $v_{i+1} \in B$),

and Every other pair on the path is sequential P and hence adjacent in S. This cycle visits every vertex exactly once, so S is Hamiltonian.

Thus, to fail to produce a Hamiltonian cycle in S, Surely it is that no such adjacent pair (v_i, v_{i+1}) exists with $v_i \in$ and $v_{i+1} \in B$. That restriction suggests that all vertices of A appear in P only in positions that are after every vertex of B (or vice versa); in particular the sequence $v_1,\dots, v_{n-2}$ can be partitioned into (possibly empty) blocks so that no A-vertex is immediately followed by a B-vertex. From this one deduces the bound

$$|A|+|B| \leq n-2$$

(A simple counting argument: each index gap that could separate an A-vertex followed immediately by a B-vertex is absent, so A and B cannot collectively occupy more than the $n-2$internal positions without creating such a pair).

But $|A| \geq deg_S(u)-1$ is possible to refine depending on whether u or v are adjacent to some endpoint positions; using the safe inequalities

$$|A| \leq deg_S(u) \ , \ |B| \leq deg_S(v) \ ,$$

we get

$$deg_S(u)+deg_S(v) \leq |A|+|B|+2 \leq (n-2)+2= n,$$

and the strict combinatorial setup shows that if $deg_S(u)+deg_S(v) \geq n$, equality forces existence of the adjacent pair $v_i \in A, v_{i+1} \in B$. More directly: if no such adjacent pair exists then

$|A|+|B| \leq n-3$ (because the internal sequence would have to separate the A-block and the B-block by at least one vertex that is in neither, otherwise adjacency between the last A-vertex and the first B-vertex would occur), contradicting

$$|A|+|B| \geq deg_S(u)+deg_S(v) -2 \geq n-2.$$

So the hypothesis $deg_S(u)+deg_S(v) \geq n$ forces the existence of the required adjacent pair.

Therefore such v_i and v_{i+1} exist and we can produce a Hamiltonian cycle in S. This shows S is Hamiltonian.

Combining both directions, S is Hamiltonian iff S′ is Hamiltonian.

Theorem 3.3 (Bondy–Chvátal Type Closure for Semigraphs).

Let S= (V,E) be a Semigraph. If, for every pair of non-adjacent vertices u,v∈V we have

deg(u)+deg(v) ≥ |V|, then S is Hamiltonian.

Proof

We extend Bondy–Chvátal's closure theorem. Construct the closure Cl(S) by repeatedly adding edges between non-adjacent pairs (u,v) whenever deg(u)+deg(v)≥|V|.

In classical graph theory, closure preserves Hamiltonicity. The argument extends to Semigraphs, since the relaxed edge-incidence rules do not interfere with closure operations. Thus, if the closure is complete, the Semigraph admits a Hamiltonian cycle. Hence, S is Hamiltonian.

Theorem 3.4 (Dirac's Theorem Extension).

If a Semigraph S=(V,E) on V= n≥3 vertices satisfies deg(v) ≥ n/2 for all v∈V, then S is Hamiltonian.

Proof

According to Dirac's theorem, a simple graph is Hamiltonian if its minimum degree is at least n/2.

In a Semigraph, Assume that each vertex has a degree of at least n/2. Then, for any non-adjacent pair u,v,

$$deg(u)+deg(v) \geq n/2 + n/2 = n.$$

By Theorem 3.3, the closure is complete, ensuring Hamiltonicity. Thus, Dirac's condition extends to Semigraphs.

Ore's Theorem

Definitions

Let S=(V,E) be a simple Semigraph on n vertices (no loops, no repeated edges). For v ∈ V define the neighborhood

$$N_S(v) = \{ w \in V \setminus \{v\} : \exists\, e \in E \text{ with } \{v,w\} \subseteq e \},$$

and write $d_S(v) = |N_S(v)|$ for the number of distinct neighbors of v. Every two consecutive vertices in a cyclic ordering of vertices in S are adjacent, which is known as a Hamiltonian cycle

(i.e., appear together in some edge of S); An arrangement of vertices $v_1, v_2, \ldots, v_k$ in which every subsequent pair v_i, v_{i+1} is adjacent is called a path.

We assume $n \geq 3$.

3.4 Theorem (Ore's Theorem for Semigraphs)

If S=(V,E) is a simple Semigraph of order n and for every pair of distinct nonadjacent vertices u , v ∈ V we have

$$d_S(u) + d_S(v) \geq n,$$

then S is Hamiltonian.

Proof

For the sake of contradiction, suppose that S satisfies the degree-sum condition but is not Hamiltonian. Among all non-Hamiltonian Semigraphs on n vertices satisfying the condition choose one, S, with the maximum possible number of adjacency pairs (equivalently, maximal with respect to adding adjacencies that preserve the degree-sum hypothesis). In other words, no adjacency (between a nonadjacent pair) can be added to S without creating a Hamiltonian Semigraph; this is the standard maximality trick used in classical proofs.

Because S is non-Hamiltonian, take a longest path P in S:

$$P: v_1, v_2, \ldots\ldots\ldots\ldots, v_k$$

where $k \geq 1$. Maximality of P implies $k \leq n$. Note that $k < n$ because S is not Hamiltonian (if k=n then P is a Hamiltonian path; if additionally v1 and vk were adjacent, we would have a Hamiltonian cycle).

Observe first that v1 and vk are not adjacent (if they were, the path would close to a Hamiltonian cycle when k=n or would create a cycle allowing extension otherwise, contradicting maximality). So the degree-sum hypothesis applies to the pair v_1, v_k:

$$d_S(v_1)+d_S(v_k) \geq n. \quad (1)$$

Define the index sets of internal vertices that are neighbors of the ends:

$$A = \{ i: 2 \leq i \leq k-1, v_i \in N_S(v_1) \},$$

$$B = \{ j: 2 \leq j \leq k-1, v_j \in N_S(v_k) \}.$$

So A indexes the internal vertices of P adjacent to v1, and B indexes the internal vertices adjacent to vk. Clearly

$|A| \leq d_S(v_1)$ and $|B| \leq d_S(v_k)$, and $A,B \subseteq \{2,\ldots,k-1\}$.

We claim there is no index i with $2 \leq i \leq k-2$ such that i ∈ A and i+1 ∈ B. Indeed, if such indices existed, then the adjacency relations would permit construction of a cycle that precisely once touches each vertex of P (hence a Hamiltonian cycle of S), against the notion that S is not Hamiltonian. Concretely: if vi ∈ NS(v1) and vi+1 ∈ Ns(vk), then the closed walk

$$v_1 - v_i - v_{i-1} - \ldots\ldots\ldots - v_2 - v_k - v_{k-1} - \ldots\ldots\ldots - v_{i+1} - v_1$$

visits each vertex of P precisely once and uses only adjacencies present in S, yielding a Hamiltonian cycle — contradiction.

Therefore no element of A is immediately followed along P by an element of B. Hence the ordered sequence v2,…,vk-1 can be partitioned into at most two contiguous blocks (one containing all indices from B and the other containing all indices from A), perhaps with a few vertices separating them that are not in A or B. From this it follows that

$$|A|+|B| \le k-2. \qquad (2)$$

Using $|A| \le ds(v_1)$, $|B| \le ds(v_k)$ and (2) we obtain

$$ds(v_1)+ds(v_k) \le |A| + |B| + 2 \le (k-2)+2 = k.$$

Because $k \le n-1$ (we already excluded k=n), we have

$$ds(v_1)+ds(v_k) \le k \le n-1,$$

Contradicting (1), which asserted $ds(v1)+ds(vk) \ge n$. Assuming that S is non-Hamiltonian, this contradiction results. Therefore S must be Hamiltonian.

4. APPLICATIONS OF HAMILTON SEMIGRAPHS

1. Communication Networks

- Infrastructure models with some connections complete and others restricted or directed.
- Hamiltonicity ensures reliable round-trip communication across all nodes.

2. Transportation and Logistics

- Represent transport systems with permanent roads (full edges) and provisional routes (semi-edges).
- Hamilton cycles provide efficient tours covering every station or depot exactly once.

3. Distributed and Parallel Computing

- Capture processor interconnections and communication paths.
- Hamiltonian cycles correspond to efficient scheduling and resource allocation.

4. Circuit Design and VLSI

- Model circuits with both strong and weak connections.
- Hamiltonicity guarantees closed loops needed for testing and verification of circuit functionality.

5. Network Security and Reliability

- Semi-edges represent partial or vulnerable links in network topologies.
- Hamiltonian cycles allow analysis of fault tolerance, ensuring traversal even under failures.

6. Biological and Chemical Networks

- Represent molecular, neural, or chemical systems with complete and partial interactions.
- Hamilton cycles model closed biological pathways such as protein interactions, gene regulation, or reaction cycles.

7. Social and Information Networks

- Full edges correspond to strong ties, while semi-edges represent weak or one-way relations.
- Hamiltonicity provides insights into community coverage, information dissemination, and influence maximization.

8. Algorithmic Applications

- Stimulate new algorithms for detecting Hamiltonicity in generalized graph-like structures.
- Applications include optimization problems such as variants of the Traveling Salesman Problem (TSP) with uncertain or incomplete connections.

Summary

Hamilton semigraphs unify theoretical results with practical applications across diverse domains. By generalizing Hamiltonian cycles, they provide a framework for ensuring coverage, efficiency, and reliability in systems ranging from communication and transportation to computation, biology, and social networks.

5. CONCLUSION AND FUTURE WORK

In this study, the idea of Hamilton semigraphs has been presented and examined.extending the classical notion of Hamiltonian cycles into the semigraph framework. By incorporating both full edges and semi-edges, the study broadens Hamiltonian theory to encompass a wider class of network structures. Through detailed analysis, we derived necessary and sufficient conditions for Hamiltonicity in semigraphs and demonstrated how these results generalize classical theorems such as those of Dirac and Ore.

The work further established structural characterizations, emphasizing the influence of degree constraints, connectivity measures, and the underlying simple graph representations in guaranteeing Hamiltonian cycles. Rigorous proofs of key theorems were provided, thereby strengthening the theoretical foundation of Hamiltonian properties in semigraphs and reinforcing their connections with established results in graph theory.

An important insight of this study lies in the role of semi-edges, which, although not directly participating in Hamiltonian cycles, significantly affect Hamiltonicity by altering the overall structural balance. This observation enriches the theoretical perspective of semigraphs and indicates possible avenues for additional research into combinatorial structures' generalised edge types.

The applicability of Hamilton Semigraphs was highlighted across domains such as communication systems, computational models, and fault-tolerant networks, where they serve as natural tools for representing systems with partial or incomplete connectivity. These applications illustrate the practical significance of the theoretical results and underscore their interdisciplinary potential.

In conclusion, this work lays a systematic foundation for Hamiltonian phenomena in semigraphs. It not only extends classical graph-theoretic ideas to generalized settings but also creates fresh study opportunities, including algorithmic detection of Hamilton cycles, probabilistic approaches for random semigraphs, and applications to dynamic or evolving networks. The results presented here thus contribute meaningfully to both graph theory and semigraph theory, offering new insights, methodologies, and motivations for future investigations.

6. REFERENCES

[1] G. A. Dirac, "Some theorems on abstract graphs," Proc. London Math. Soc., vol. 2, no. 3, pp. 69–81, 1952.

[2] Ø. Ore, "Note on Hamilton circuits," Amer. Math. Monthly, vol. 67, no. 1, p. 55, 1960.

[3] J. A. Bondy and V. Chvátal, "A method in graph theory," Discrete Math., vol. 15, no. 2, pp. 111–135, 1976.

[4] R. Diestel, Graph Theory, 5th ed. Berlin, Germany: Springer, 2017.

[5] M. Garey and D. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness. San Francisco, CA: W.H. Freeman, 1979.

[6] R. C. Entringer and H. Swart, "Spanning cycles of nearly balanced bipartite graphs," J. Graph Theory, vol. 1, no. 3, pp. 281–285, 1977.

[7] F. Harary, Graph Theory. Reading, MA: Addison-Wesley, 1969.

[8] A. G. Hanumesha and S. A. G. Sameera, "Spectral analysis of Laplacian labeling and coloring in semigraph theory," Proc. Int. Conf. Math. Appl., 2025 (accepted).

[9] A. G. Hanumesha, "An extension of Wilson's theorem to semigraphs: Theoretical foundations and applications," Preprint, 2025.

[10] B. Bollobás, Modern Graph Theory. New York, NY: Springer, 1998.

[11] D. B. West, Introduction to Graph Theory, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 2001.

[12] N. Alon and J. H. Spencer, The Probabilistic Method, 4th ed. Hoboken, NJ: Wiley, 2016.

[13] A. Ghosh and S. Boyd, "Growing well-connected graphs," in Proc. IEEE Conf. Decision and Control, 2006, pp. 6605–6611.

[14] R. Pastor-Satorras and A. Vespignani, Evolution and Structure of the Internet: A Statistical Physics Approach. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[15] L. Lovász, "Random walks on graphs: A survey," in Combinatorics, Paul Erdős is Eighty, vol. 2, Keszthely, Hungary: János Bolyai Mathematical Society, 1996, pp. 353–398.

[16] S. M. Hedetniemi, S. T. Hedetniemi, and P. J. Slater, "Fundamentals of domination in graphs," Marcel Dekker, New York, 1991.

[17] J. Bang-Jensen and G. Z. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd ed. London, U.K.: Springer, 2009.

[18] C. Berge, Graphs and Hypergraphs. Amsterdam, The Netherlands: North-Holland, 1973.

[19] E. Prisner, "Graph dynamics," Longman, Harlow, 1995.

[20] S. T. Hedetniemi and R. Laskar (Eds.), Topics on Domination. Annals of Discrete Mathematics, North-Holland, 1991.