On Neuotrosophic Generalized Topological Simple Groups

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Abstract
In this paper, we introduce the concept of *Neutrosophic Generalized Topological Simple Groups* (NGTSGs), synthesizing the theories of neutrosophy, topology, and abstract algebra. An NGTSG is defined as a group equipped with a neutrosophic generalized topology that satisfies the condition of simplicity, namely the absence of non-trivial neutrosophic normal subgroups. The neutrosophic framework, based on triplet-valued membership functions (truth, indeterminacy, and falsity), provides a powerful mechanism for representing and reasoning with indeterminate, inconsistent, and incomplete information. We formally define the structure and axioms of NGTSGs, examine their algebraic and topological properties, and analyze the interplay between neutrosophic topology and group operations. Conditions for continuity, separation, and connectedness in this generalized setting are investigated. Beyond theoretical contributions, we explore potential applications of NGTSGs in **artificial intelligence**, particularly in domains where uncertainty and vagueness are intrinsic, such as decision-making systems, knowledge representation, natural language understanding, and machine learning models that integrate fuzzy and quantum-inspired logic. Illustrative examples are provided to demonstrate these applications, highlighting how NGTSGs can enhance robustness and interpretability in AI-driven environments. This work establishes a foundational framework for further research in neutrosophic algebraic topology with significant interdisciplinary relevance to AI, computational intelligence, and quantum computing.

Keywords: μ_N- topological simple group, μ_N–homeomorphism.

1.INTRODUCTION
The concept of neutrosophic sets was first introduced by Smarandache [5] as an extension of intuitionistic fuzzy sets, allowing the simultaneous representation of truth, indeterminacy, and falsity. Building on this foundation, A.A. Salama and S.A. Albowi [3] formulated the notion of neutrosophic topological spaces. Later, N. Raksha Ben and G. Hari Siva Annam proposed the framework of neutrosophic generalized topological spaces [2], while C. Selvi and R. Selvi advanced the study further by introducing generalized topological simple groups [1].

2.PRELIMINARIES

Definition 2.1(Neutrosophic set):

Let X be a nonempty fixed set. A neutrosophicset(NS for short) A is an object is having the form

$$A = \{(x, \mu_A(x), \sigma_B(x), \nu_A(x)\}$$

Where $\mu_A(x)\sigma_A(x)\nu_A(x)$ which represents the degree of membership function,the degree intedeterminancy and the degree of non-membership function respectively of each element$x \in X$ to the set A. **[2]**

Definition 2.2(Neutrosophic Universal set): If $\mu_A(x) = 1, \sigma_A(x) = 0, \nu_A(x) = 0$ for all$x \in X$. Then A is said to be a universal set . It is denoted by $\tilde{X}$.**[2]**

Definition 2.3(Neutrosophic Empty set): $If\ \mu_A(x) = 0, \sigma_A(x) = 1$ and$\nu_A(x) = 1$ for all $x \in X$. Then A is said to be Neutrosophic Empty set and denoted by $\tilde{\phi}$. **[2]**

Definition 2.4(Neutrosophic Union): Neutrosophic union of A and B ,denoted by $A \cup B$ and defined by

$$A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \wedge \nu_B(x): x \in X\}.\textbf{[2]}$$

Definition 2.5(Neutrosophic Intersection): Neutrosophic intersection of A and B ,denoted by $A \cup B$ and defined by

$$A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \vee \nu_B(x): x \in X\}.\textbf{[2]}$$

Definition 2.6(Neutrosophic Product): Let X and Y be nonempty neutrosophic sets and

A=$(x, \mu_A(x), \sigma_B(x), \nu_A(x))$

B=$(x, \mu_A(y), \sigma_B(y), \nu_A(y))$ Neutrosophic sets on X and Y

$(A \times B)(x, y) = ((x, y)\ min(\mu_A(x), \mu_B(y)), \min(\sigma_A(x), \sigma_B(y)), \min(\nu_A(x), \nu_B(y)))$. **[2]**

Definition 2.7(Neutrosophictopololoy or μ_N-topology): A μ_N topology is a non - empty set X is a family of neutrosophic subsets in X satisfying the following axioms:

$(i).\ \tilde{\phi} \in \mu_N$

(ii).G1 ∪G2 $\in \mu_N$ for any G1,G2$\in \mu_N$. Throughout this article, the pair of(X,μ_N) is known as μ_N-Topological Space[μ_NTS for short]. **[2]**

Definition 2.8: The elements of $\mu_N$areμ_N −open sets and their complement is called μ_N-closed sets. **[2]**

Definition: 2.9 A $\mathcal{G}$-topological simple group G is a simple group which is also a$\mathcal{G}$-topological space if the following conditions are satisfied.

*(i).*The multiplication mapping $m : G \times G \longrightarrow G$ defined by $m(x,y) = x * y,\ x,y \in G$ is $\mathcal{G}$-continous.

(ii). The inverse mapping$i: G \longrightarrow G$ defined by $i(x) = x^{-1}, x \in G$ is $\mathcal{G}$- continous**[1]**.

3.NEUTROSOPHIC GENERALIZED TOPOLOGICAL SIMPLE GROUPS

Definition: 3.1 A μ_N-topological simple group G is a simple group which is also a μ_N-topological space if the following conditions are satisfied.

*(i).*The multiplication mapping $m : G \times G \longrightarrow G$ defined by $m(x,y) = x * y,\ x,y \in G$ is μ_N-continous.

(ii). The inverse mapping$i: G \longrightarrow G$ defined by $i(x) = x^{-1}, x \in G$ is μ_N- continous.

Example: 3.2

Let $G = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ be trivial simple group under multiplication and we can define a Neutrosophic set and $\mu_N - topological\ simple\ group$ for this ,

$$A = \{(0.5,0.4,0.3)\}$$

$$\mu_N = \{\tilde{\phi}, A, \check{X}\}$$

Now $G \times G = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right\}$

and

$$\mu_N \times \mu_N = \{\tilde{\phi} \times \tilde{\phi}, \tilde{\phi} \times A, \tilde{\phi} \times \check{X}, A \times \tilde{\varphi}, A \times A, A \times \tilde{X}, \tilde{X} \times A, \tilde{X} \times \widetilde{X,}\ \tilde{X} \times \tilde{\phi}\}$$

Clearly the multiplication mapping and inversion mapping are $\mu_N - continous$. Therefore $(G.*, \mu_N)$ Neutrosophic generalized topological simple group.

Remark: 3.3μ_N- Topological simple group⇒ Topological group, but converse need not be true.

Note: 3.4$(i). GL_n(F) = \{A \in M_n(F) : \det(A) \neq 0\}$.

$(ii). SL_n(F) = \{ A \in GL_n(F) : \det(A) = 1\}$**.**

Example: 3.5$GL_n(F)$ is a μ_N-topological group. But it is not μ_N-topological simple group,because $SL_n(F)$ is a normal subgroup of $GL_n(F)$

Example: 3.6 $SL_n(F)$ is a μ_N-topological simple group.

Example: 3.7$Z_p, (where\ p\ is\ prime)$ is μ_N-topological simple group with indiscerete μ_N-topology.

Proposition:3.8 Let G be a μ_N-topological simple group and μ_e be the collection of all μ_N-open neighbourhoods of e. Then the following holdshood.

*(i).*For every $U \in \mu_e$, there is an element $V \in \mu_e$ such that $V^{-1} \subset U$.

*(ii).*For every $U \in \mu_e$ and for every $x \in U$, there is an element $V \in \mu_e$ such that $Vx \subset U$.

Proposition: 3.9 Everyμ_N-topological simple group G has μ_N-open neighborhood at the identity element e_G consisting of symmetric μ_N-neighbourhoods.

Proof: For an arbitrary μ_N-open neighbourhood U of the identity e_G, if $V = U \cap U^{-1}$, then $V = V^{-1}$, the set V is an $\mathcal{G}$-open neighbourhood of e_G , which implies that V is a symmetric μ_N-neighbourhood and $V \subset U$.

Proposition: 3.10 Let G be aμ_N-topological simple group. Every μ_N-neighbourhood U of $e$contains an μ_N-open symmetric neighbourhood V of e such that VV$\subset U$.

Proof: Let U' be the interior of U. Consider the multiplication mapping $\mu: U' \times U' \to G$. Since μ is μ_N-continous, $\mu^{-1}(U')$ is μ_N-open and contains (e, e). Hence there are μ_N-open sets $V_1, V_2 \subset U$ such that$(e, e) \in V_1 \times V_2$, and$V_1 V_2 \subset U$. If we let $V_3 = V_1 \cap V_2$, then $V_3 V_3 \subset U$ and $V_3$is an μ_N-open neighbourhood of e. Finally, let $V = V_3 \cap {V_3}^{-1}$, which is μ_N-open, contains e and V is symmetric and satisfies $VV \subset U$.

Corrollary: **3.11** Let G be μ_N-topological simple group.Every μ_N-neighbourhood of e_G contains an μ_N-open symmetric neighbourhood$V$ of $e_G$such thatthat$VV^{-1} \subset U$ and $V^{-1}V \subset U$.

Proof: Since V is a symmetric μ_N-open neighbourhood ,$V = V^{-1}$. Therefore $VV^{-1} \subset U$and$V^{-1}V \subset U$.

Proposition: 3.12 Let $f: SL_n(F)$ to $SL_n(F)$ be a μ_N-homeomorhism.Then

(i). $A \to A^{-1}$

(ii).$A \to \overline{A}$

(iii). $A \to A^T$

(iv). $A \to A^*$, *whereh* A^* *is the conjugate tranpose.*

Proposition: 3.13 Let G beμ_N-topological simple group. Then the following maps are μ_N-homeomoephism from G to G for all $g \in G$.

*(i).*The right(left) translation $l_g(r_g)$ of G by g is a $\mathcal{G}$-homeomorphism of the space G onto itself.

(ii). The inverse map :$x \to x^{-1}$.

(iii). The inner automorphismmap **:**$x \to gxg^{-1}$.

Corrollary: 3.14 Let G beμ_N-topological simple group and U be $\mathcal{G}$-open subset of G , F is μ_N-closed in G and A be any subset of G.Then

(i). aU,Ua and AU, UA are $\mathcal{G}$-open in G.

(ii). aFandFa are μ_N-closed in G.

Corollary: 3.15 Let μ_e be a collection of allμ_N-open sets of G at e. Then $\mu_g = \{ Ug: U \in \mu_e\}$ is also a collection of μ_N-open sets at g.

Corollary: 3.16 Suppose that a subgroup of $\mathcal{G}$-topological simple group G contains a non-empty μ_N-open subset of G. Then H isμ_N-open in G.

Proof: Let U be a μ_N-open non-empty subset of G with $U \subset H$. For every $a \in H$, By corollary 3.15, the set $l_g(a) = Ua$ μ_N-open in G, then $H = \cup_{a\in H} Ua$ is μ_N-open in G.

Proposition: 3.17 Let $f: G \to H$ be a homomorphism of μ_N-topological simple groups.If f is μ_N-continous atthe neutral element e_G of G, then f is μ_N-continous.

Proof: Let $x \in G$ be arbitrary and suppose that W is an μ_N-open neighbourhood of $y = f(x)$in H.Since the left translationl_y is a μ_N-homeomorphism of H, there exists an μ_N-openneighbourhood$V$ of the neutral element e_H in H such that$l_y(V) = yV$ is an μ_N-open neighbourhood of y. Then $yV \subseteq W$. Since f is μ_N-continous at $e_G$of G, then $f(U) \subset V$, for some μ_N-open neighbourhood U of e_G in G. sincel_x is a homeomorphism of G onto itself, then xU is an μ_N-open neighbourhood of x in G.

Now we have$f(xU) = f(x)f(U)$

$= yf(U)$

$\subseteq yU$

$\subseteq W$. Hence f is μ_N-continous at the point $x \in G$.

Proposition: 3.18$SL_n(F)$ is an μ_N-open subset of $M_n(F)$.

Proof: Let A be an element of $M_n(F)$ and let $f(A) = \det(A)$ be a function from $M_n(F)$ to F. Since $SL_n(F)$ contains the matrices of $M_n(F)$ with determinant 1.Then $SL_n(F)$= $M_n(F)\backslash B$, where B is a matrix with $\det(B) \neq 1$. Since the determinant function is a polynomials and polynomials are μ_N-continous,f is μ_N-continous. Since $\{0\}$ is $\mathcal{G}$-closed in F, then $f^{-1}(\{0\})$ is also μ_N-closed in $M_n(F)$. Hence $SL_n(F)$ isμ_N-open.

Proposition: 3.19 Suppose that G, H and K are μ_N-topological simple groups and that ϕ:$G \to H$ andψ: $G \to K$ are homomorphism Such that $\psi(G) = K$ and $Ker\psi \subset Ker\phi$. Then there exists homomorphism $f: K \to H$ such that $\phi = fo\psi$. In addition, for each $\mathcal{G}$-neighbourhood$U$ of the identity elemente_H in H, there exists a $\mathcal{G}$-neighbouhood$V$ of the identity element e_k in K such that ψ^{-1}(V) $\subset$ ϕ^{-1}(U), then f is μ_N-continous.

Proof: Suppose $U$is μ_N-neighbourhood of e_H in H. By assumption, there exists aμ_N-neighbouhood$V$ of the identity element e_k in K such that ,$W = \psi^{-1}$(V) $\subset$ ϕ^{-1}(U).

$\Rightarrow \phi(W) = \varphi(\psi^{-1}$(V)) $\subset$ $\phi(\phi^{-1}$(U))

$\Rightarrow \phi(W) = f(V) \subset$ U

$\Rightarrow \phi(W) \subset$ U. Hence f is μ_N-continous at the identity element of K. Therefore f is μ_N-continous.

Corollary: 3.20 Let$\phi: G \to H$ and$\psi: G \to K$ be μ_N-continous homomorphism of a μ_N-topological simple groups G, H and KSuch that$\psi(G) = K$ and $Ker\psi \subset Ker\ \phi$. If the homomorphism ψ is $\mathcal{G}$-open, then there exists a μ_N-continoushomomorhism, $f: K \to H$ such that $\phi = fo\psi$.

Proof: The existence of a homomorphism $f: K \to H$ such that $\phi = f\ o\ \psi$.Take an arbitrary μ_N-open set Vin H. Then $f^{-1}(V) = \psi(\phi^{-1}(V))$. Since$\phi$ is μ_N-continous and ψ is an μ_N-open map, $f^{-1}(V)$ is μ_N-open in K. Therefore f is μ_N-continous.

Proposition: 3.21 Let G and H be μ_N-topological simple groups with neutral element e_G and e_H, respectively, and let p be aμ_N-continous homomorphism of G onto H such that, for some non-empty subset$U$ of G, the set p(U) is μ_N-open in H and the restriction of p to U is an μ_N-open mapping of U onto $p(U)$. Then the homomorphism p is μ_N-open.

Proof: It suffices to show that $x \in G$, where W is an μ_N-open neighbourhood of x in G, then $P(W)$is aμ_N- open neighbourhood of $p(x)$ in H. Fix a point $y$in U, and let l be the left translation of G byyx^{-1}. Then l is a homomorphism of G onto itself such that ,

$$l_{yx^{-1}}(x) = yx^{-1}x$$

$$= y.$$

So $V = U \cap l(W)$ is an μ_N-open neighbourhood of $y$in U. Then $p(V)$ is μ_N-open subset of H. consider the left translation h of H by the inverse to $p(\text{y}x^{-1})$.

Now clearly, $(h\ o\ p\ o\ l)(x) = h(p(l(x)))$

$$= h(p(y))$$

$$= p(xy^{-1})p(y)$$

$$= p(xy^{-1}y)$$

$$= p(x).$$

Hence $h\left(p\left(l(W)\right)\right) = p(W)$. Clearly h is aμ_N- homeomorphism of H onto itself. Since $p(V)$ is μ_N-open in H, $h(p(V))$ is also μ_N-open in H. Therefore $p(W)$ contains the μ_N-open neighbourhood $h(p(V))$ of $p(x)$in H. Hence $p(W)$ is a μ_N-open neighbourhood of $p(x)$ in H.

Proposition: 3.22 Let G be μ_N-topological simple group. Then

(i). If H is a subgroup of G, then $\overline{H}$ also a subgroup of G.

*(ii)***.** If H is a normal subgroup of G, $\overline{H}$ also a normal subgroup of G.

Proof: *(i).*Forsubgroup we have to prove the following conditions.

(a). $\overline{H}$ is non empty.

(b). $x, y \in \overline{H} \Rightarrow xy \in \overline{H}$

(c). $x \in \overline{H} \Rightarrow x^{-1} \in \overline{H}$

(a). Since $e \in H \Rightarrow e \in \overline{H}$. So $\overline{H}$ is non empty.

(b). Let $g, h \in \overline{H}$. Let U be an μ_N-open neighbourhood of gh. Let $\mu : G \times G \to G$ denote the multiplication map which is $\mathcal{G}$-continous. Then $\mu^{-1}(U)$ is $\mathcal{G}$-open in $G \times G$ and contains (g, h). so,there are μ_N-neighbourhoodV_1 of g and V_2 of h such that $V_1 \times V_2 \subset \mu^{-1}(U)$. Since$g, h \in \overline{H}$, then there are points $x \in V_1 \cap H \neq \phi$ and $y \in V_2 \cap H \neq \phi$. Since $x, y \in H, xy \in H$ and since $(x, y) \in \mu^{-1}(U)$, then $xy \in U$. Thus $xy \in U \cap H \neq \phi$. Since U is an arbitrary μ_N-open neighbourhood of gh, then we have $gh \in \overline{H}$.

(c). Now let $i: G \to G$ denote the inverse map, and W be an μ_N-open neighbourhood of h^{-1}. Then $i^{-1}(W) = W^{-1}$isμ_N-open and contains h, so there is a point $z \in H \cap W^{-1} \neq \phi$. Then we have $z^{-1} \in H \cap W \neq \phi$. Therefore $h^{-1} \in \overline{H}$.

(ii). Now we have to prove that $g\overline{H}g^{-1} \in \overline{H} \forall g \in G$.

Since H is a normal subgroup of G, $gHg^{-1} \in H \forall g \in G$.

Now $\overline{gHg^{-1}} \subset \overline{H} \forall g \in G$.

$$\Rightarrow g\overline{H}g^{-1} \subset \overline{H} \forall g \in G.$$

$\Rightarrow g\overline{H}g^{-1} \in \overline{H}, \forall g \in G$. Therefore $\overline{H}$ is a normal subgroup of G.

Corollary:3.23 Let$G$ be μ_N-topological simple groupand H be the centre of a Hausdorffμ_N-topological simple group$G$. Then $\overline{H}$ is asubgroup of G.

Corollary: 3.24 Let G be μ_N-topological simple group$Z(G)$ be the centre of G. Then$\overline{Z(G)}$ is a normal subgroup of G.

Proof:proof follows from the above theorem.

Corollary: 3.25 Let G and Hbeμ_N-topological simple groups.If $f: G \to H$ is a homomorphismmapping,then$\overline{ker f}$ is a normal subgroup of G.

4.Conclusion

Beyond their theoretical foundation, NGTSGs are particularly relevant to **artificial intelligence**, where handling uncertainty, vagueness, and incomplete knowledge is essential. In AI, neutrosophic models have proven useful in areas such as decision-making, expert systems, natural language processing, and intelligent reasoning under ambiguity. NGTSGs provide a structured mathematical framework that can enhance knowledge representation, improve the robustness of learning algorithms, and strengthen interpretability in systems that rely on uncertain or conflicting data.

Thus, this work contributes not only to neutrosophic algebraic topology but also to its emerging role in AI, offering new perspectives for computational intelligence, knowledge engineering, and machine learning in uncertain environments.

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