The Rigidity of Minimal Translation-Factorable Surfaces in Hyperbolic 3-Space

Sid Ahmed Difi[1], Zouaoui Bekri[2,3,*] and Abdellatif Berbiche[4]

[1]Université des Sciences et de la Technologie d'Oran, Faculté des Mathématiques et Informatique, Département de Mathématiques, Oran, Algeria
sidahmed.difi@yuniv-usto.dz
[2]Laboratory of Fundamental and Applied Mathematics, University of Oran 1, Ahmed Ben Bella, Es-Senia 31000, Algeria
[3,*]Department of Sciences and Technology, Institute of Sciences, Nour-Bachir University Center, El-Bayadh 32000, Algeria
zouaouizargui22@gmail.com
[4]national school of agronomy, Algier, Algeria
abdellatif.berbiche@edu.ensa.dz

Abstract

This paper presents a complete classification of minimal translation-factorable surfaces of type I in hyperbolic 3-space, $\mathbf{H}^3$. We work in the upper half-space model $\mathbf{R}^3_+ = \{(u, v, z) \in \mathbf{R}^3 \mid z > 0\}$ endowed with the metric $ds^2 = (du^2 + dv^2 + dz^2)/z^2$. The parametrization defines the surfaces under investigation

$$\mathbf{r}(u, v) = (u, v, \zeta(u)(\eta(v) + c) + c\eta(v)),$$

where ζ and η are smooth functions and c is a real constant. By computing the fundamental forms and the mean curvature derived from the hyperbolic metric, we prove that the only minimal surfaces in this class are the totally geodesic planes. Our method involves a detailed analysis of the partial differential equation implied by the vanishing mean curvature condition, $H = 0$, showing that any assumption of non-constant ζ or η leads to a contradiction. This rigidity result highlights the profound geometric constraint imposed by the negative curvature of $\mathbf{H}^3$. It extends the classical classification schemes from Euclidean and Lorentz-Minkowski spaces to the hyperbolic setting.

Keywords: Hyperbolic space, Minimal surfaces, Mean curvature, Lorentz-Minkowski,Translation-Factorable (TF) surfaces, Laplace operator.
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1 Introduction

The theory of minimal surfaces stands as one of the most elegant and deeply studied chapters in differential geometry, with its origins tracing back to Euler and Lagrange's work on the calculus of variations in the 18th century. A minimal surface is formally defined as a surface that locally minimizes its area, a condition that is geometrically equivalent to the vanishing of its mean curvature at every point. In the familiar Euclidean space ^{3}, this field has flourished, yielding a rich catalogue of classical examples such as the catenoid, the helicoid, and the renowned Plateau's problem, which concerns the existence of minimal surfaces with given boundaries. The exploration of this concept has naturally expanded beyond Euclidean geometry into spaces endowed with different metrics and curvatures. A significant body of work has been devoted to the study of surfaces in the Lorentz-Minkowski space $\mathbf{E}^3_1$. In this setting, a prominent research direction involves characterizing surfaces through the condition $x = Ax + B$, where is the Laplace operator, a line of inquiry initiated by Alías, Ferrandez, and Lucas [1]. This framework has been successfully applied to various families of surfaces. For instance, Baba-Hamed, Bekkar, and Zoubir investigated both helicoidal [3] and translation [4] surfaces of revolution satisfying this condition and its variants, such as $r_i = \lambda_i r_i$. Similarly, Bekkar and Zoubir [6] and later Senoussi and Bekkar [18] extended this analysis to other helical and revolution surfaces, uncovering their unique properties in a Lorentzian context. The concept of translation and factorable surfaces offers a distinct and powerful approach for constructing and classifying surfaces. These surfaces are defined by a parametrization that can be expressed as a sum or product of functions depending on a single variable, effectively "translating" or "factoring" generating curves. Their decomposable nature often simplifies the associated curvature equations, making them fertile ground for classification. The classification program for such surfaces has been vigorously pursued across various geometric ambient spaces. In the pseudo-Galilean space, Aydin, Ogrenmis, and Ergut provided a complete classification of factorable surfaces [2]. In Euclidean space, the works of Liu [13, 14] and later López and Moruz [16] have been instrumental in characterizing translation surfaces with constant mean curvature or constant Gaussian curvature. The factorable minimal surfaces were also explored by Yu and Liu [20], while Meng and Liu [17] extended this study to the Minkowski 3-space. A pivotal concept that connects many of these studies is that of surfaces of finite type, where the coordinate functions are eigenfunctions of the Laplace operator. The foundational texts by Chen [7, 8] and subsequent work by Garay [12] and Dillen, Pas, and Verstraelen [11] have deeply explored this connection, providing a unifying algebraic framework for understanding a wide range of surface families. The geometry of the Gauss map for specific surfaces, such as helicoidal surfaces studied by Balkoussis and Verstraelen [5] and surfaces of revolution in Minkowski space analyzed by Choi [9], further enriches this tapestry. Despite the extensive literature in Euclidean and Lorentzian settings, the study of minimal surfaces in hyperbolic space $\mathbf{Hsp}^3$ presents a fundamentally different and more rigid landscape. The constant negative curvature of $\mathbf{Hsp}^3$ imposes strong restrictions, often limiting the diversity of minimal surfaces. López [15] made a significant contribution by proving that minimal translation surfaces in $\mathbf{Hsp}^3$ are necessarily totally geodesic planes, a stark contrast to the rich families found in $\mathbf{E}^3$. This highlights the profound impact of ambient curvature on the existence and classification of minimal surfaces. In this paper, we bridge and extend these lines of inquiry by conducting a comprehensive analysis of minimal translation factorable surfaces in the three-dimensional hyperbolic space $\mathbf{Hsp}^3$. We employ the half-space model, $\mathbf{R}^3_+ = \{(u,v,z) \in \mathbf{R}^3 \mid z > 0\}$ with the hyperbolic metric $ds^2 = \frac{du^2+dv^2+dz^2}{z^2}$, which provides a convenient setting for calculations. We focus on surfaces of Type I, parameterized by

$$r(u,v) = (u, v, \zeta(u)(\eta(v)+c) + c\eta(v)),$$

where ζ and η are smooth functions and c is a real constant. Building upon the foundational work of Dif, Hakem, and Zoubir on translation factorable surfaces in Euclidean and Lorentzian spaces [10], and inspired by the rigidity results of López [15], we undertake a detailed computation of the fundamental forms and the corresponding mean curvature. Our main theorem establishes that within this class, the only minimal surfaces are the totally geodesic planes. This result underscores the extreme rigidity inherent in the hyperbolic geometry. The proof proceeds via a meticulous case analysis of the governing partial differential equation derived from the condition $H = 0$. We demonstrate that any assumption leading to non-constant ζ or η ultimately results in a mathematical contradiction, thereby forcing the surface to be planar. This work thus generalizes the classification paradigms from Euclidean and Lorentz-Minkowski spaces to the hyperbolic realm, confirming the powerful constraints imposed by negative curvature and contributing a definitive result to the global theory of minimal surfaces.

The structure of this paper is organized as follows. In Section 2, we establish the foundational framework, recalling the hyperbolic half-space model and providing the precise definitions for translation-factorable surfaces of type I. Section 3 is dedicated to a detailed computation of the fundamental forms and the derivation of the mean curvature formula for these surfaces. The core of our classification result is presented in Section 4, where we conduct a comprehensive case analysis to prove that the minimality condition forces the surface to be a totally geodesic plane. Finally, in Section 5, we conclude by discussing the implications of our rigidity result and its place within the broader context of minimal surface theory in non-Euclidean spaces.

2 Preliminaries

In the preliminary part of this work, we introduce the definitions and theorem of the translation factorable surface.

Definition 2.1 *A surface M in $\mathbf{H}^3$ is translation factorable if it admits an immersion $X : U \subset \mathbf{R}^2 \to \mathbf{R}^3_+$ of the form Type I: $r(u,v) = (u, v, \zeta(u)(\eta(v)+c) + c\eta(v))$, $(u,v) \in U$*

where ζ and η are smooth functions on open subsets of $\mathbf{R}$, and c is a real constant.
Through detailed computations of the fundamental forms and curvature equations for this types, we demonstrate that the only minimal translation factorable surfaces are the totally geodesic planes. This

result highlights the rigidity imposed by minimality in the hyperbolic setting, extending analogous classifications from Euclidean and Lorentz-Minkowski spaces.

Definition 2.2 *Consider the half-space model of* $\mathbf{H^3}$. *A surface* M *in hyperbolic space* $\mathbf{H^3}$ *is a translation factorable surface if it is given by an immersion* $X : U \subset \mathbf{R^2} \to \mathbf{R^3_+}$ *written as*

$$r(u;v) = (u;v;\zeta(u)(\eta(v)+c)+c\eta(v));(u;v) \in U \ (type \quad I) \tag{1}$$

where f *and* g *are smooth functions on opens of* $\mathbf{R}$.

In Euclidean space, a minimal surface in $\mathbf{H^3}$ is a surface for which the mean curvature H is zero at every point. Examples of minimal surfaces are totally geodesic planes. In the literature, examples of minimal surfaces in hyperbolic space have been found for solving the corresponding Dirichlet problem and obtaining minimal graphs.
Usually, we locally compute the mean curvature He by the classical formula
$H_e = \dfrac{EN - 2MF + GL}{EG - F^2}$, where $\{L;M;N\}$ and $\{E;F;G\}$ are the coefficients of the second and first fundamental form of r, respectively, computed with respect to the Euclidean metric.

Theorem 2.3 *(Classification of Minimal Translation Factorable Surfaces of Type I)*

In the half-space model of $\mathbf{H^3}$, *minimal translation factorable surfaces of type I, parameterized by*

$$\mathbf{r}(u,v) = (u, v, +\zeta(u)(\eta(v)+c)+cg(v)),$$

where ζ *and* η *are smooth functions,* c *is a real constant and the mean curvature* $H=0$ *everywhere, occur precisely when* $\eta(v)=c_0$ *is a constant* ($c_0 \in \mathbb{R}$, $c_0 \neq 0$)*, and* $\zeta(u)$ *satisfies the relation*

$$u = \int \frac{\mathrm{d}\zeta}{\sqrt{-\frac{1}{c_0} + c_1\left(2(\zeta+1)c_0+1\right)^{-4/c_0}}} + c_2,$$

with constants $c_1, c_2 \in \mathbb{R}$.

3 Surfaces of type I

In this section, we consider surface in $\mathbf{E^3_1}$. Assume that $\mathbf{M^2}$ is equivalent to

$$r(u,v) = (u, \zeta(u)(\eta(v)+c)+c\eta(v), v). \tag{2}$$

The coefficients of the first fundamental form are

$$E = \zeta_u^2(\eta+c)^2+1, \quad F = \zeta_u\eta_v(\zeta+c)(\eta+c), \quad G = 1+\eta_v^2(\zeta+1)^2$$

and the coefficients of the second fundamental form are

$$L = \frac{\zeta_{uu}(\eta+c)}{\sqrt{W}}, \quad M = \frac{\zeta_u\eta_v}{\sqrt{W}}, \quad N = \frac{\eta_{vv}(\zeta+c)}{\sqrt{W}}$$

where $W = EG - F^2 = \zeta_u^2(\eta+c)^2 + \eta_v^2(\eta+c)^2 + 1$.
The Gaussian curvature K and the mean curvature H are defined respectively by
$K = \dfrac{LN - M^2}{EG - F^2}$ and $H = \dfrac{1}{2}\dfrac{EN - 2MF + GL}{EG - F^2}$.
Therefore, the mean curvature H becomes

$$H = \frac{(\zeta_u^2(\eta+c)^2+1)\eta_{vv}(\zeta+c) - 2\zeta_u^2\eta_v^2(\zeta+c)(\eta+c)}{2W^{3/2}} + \frac{(1+\eta_v^2(\zeta+c)^2)\zeta_{uu}(\eta+c)}{2W^{3/2}}. \tag{3}$$

We assume that M is a translation surface of type I given by (1).

The expressions of He and N_3 are

$$H_e = \frac{1}{2}\frac{(\zeta_u^2(\eta+c)^2+1)\eta_{vv}(\zeta+c) - 2\zeta_u^2\eta_v^2(\zeta+c)(\eta+1) + \zeta_{uu}(\eta+c)(1+\eta_v^2(\zeta+c)^2)}{\Big((\zeta_u(\eta+c))^2+(\eta_v(\zeta+c))^2+1\Big)^{3/2}}$$

and

$$N_3 = \frac{1}{\Big((\zeta_u(\eta+c))^2+(\eta_v(\zeta+c))^2+1\Big)^{1/2}}$$

respectively. If the surface is minimal, that is, $H=0$ on M.

From (3), we have

$$(\zeta(\eta+c)+c\eta)\frac{(\zeta_u^2(\eta+c)^2+1)\eta_{vv}(\zeta+c) - 2\zeta_u^2\eta_v^2(\zeta+c)(\eta+c) + \zeta_{uu}(\eta+c)(1+\eta_v^2(\zeta+c)^2)}{\Big((\zeta_u(\eta+c))^2+(\eta_v(\zeta+c))^2+1\Big)^{3/2}}$$

$$+\frac{2}{\Big((\zeta_u(\eta+c))^2+(\eta_v(\zeta+c))^2+1\Big)^{1/2}} = 0. \tag{4}$$

This equation can be written as follows

$$((\zeta+1)(\eta+c)-1)\Big(\zeta_u^2(\zeta+c)(\eta_{vv}(\eta+c)^2-\eta_v^2(\eta+c)) + \eta_{vv}(\zeta+c)\Big)$$

$$+((\zeta+1)(\eta+c)-1)\Big(\eta_v^2(\zeta+c)(\zeta_{uu}(\zeta+c)^2-\zeta_u^2(\zeta+c)) + \zeta_{uu}(\eta+c)\Big)$$

$$= -2\Big((\zeta_u(\eta+c))^2+(\eta_v(\zeta+c))^2+1\Big). \tag{5}$$

We consider $\Theta = \zeta + c$ and $\Phi = \eta + c$, we have

$$\Big(\Theta_u^2(\Phi_{vv}\Phi - \Phi_v^2) + \frac{\Phi_{vv}}{\Phi} + \Phi_v^2(\Theta_{uu}\Theta - \Theta_u^2) + \frac{\Theta_{uu}}{\Theta}\Big) + 2\Big(\frac{\Theta_u^2}{\Theta^2} + \frac{\Phi_v^2}{\Phi^2} + \frac{1}{\Theta^2\Phi^2}\Big)$$

$$= -\Big(\frac{\Theta_u^2}{\Theta}\Big(\Phi_{vv} - \frac{\Phi_v^2}{\Phi}\Big) + \frac{1}{\Theta}\frac{\Phi_{vv}}{\Phi^2} + \frac{\Phi_v^2}{\Phi}\Big(\Theta_{uu} - \frac{\Theta_u^2}{\Theta}\Big) + \frac{\Theta_{uu}}{\Theta^2}\frac{1}{\Phi}\Big). \tag{6}$$

Case A: (8) reduces to

$$\Big(\Theta^2\Theta_u^2(\Phi_{vv}\Phi^3 - \Phi^2\Phi_v^2) + \Theta^2\Phi\Phi_{vv} + \Phi^2\Phi_v^2(\Theta_{uu}\Theta^3 - \Theta^2\Theta_u^2) + \Phi^2\Theta\Theta_{uu}\Big) + 2\Big(\Phi^2\Theta_u^2 + \Theta^2\Phi_v^2 + 1\Big)$$

$$+\Theta\Theta_u^2\Big(\Phi^2\Phi_{vv} - \Phi\Phi_v^2\Big) +_{vv} + \Phi\Phi_v^2\Big(\Theta^2\Theta_{uu} - \Theta\Theta_u^2\Big) + \Phi\Theta_{uu} = 0 \tag{7}$$

We have three cases in order to solve (9):

Case A.1. $\Phi = \Phi_0 \neq 0 \in$, (10) immediately implies

$$\Theta_{uu}(\Theta\Phi_0^2 + \Phi_0) + 2G_0^2\Theta_u^2 + 2 = 0 \tag{8}$$

We replace $\Theta_u = T$ and we have the following Bernouli's equation

$$T'(\Theta G_0^2 + \Phi_0) + 2\Phi_0^2 T^2 + 2 = 0.$$

Case A.2. $\Phi = c_1 v + c_2$; $c_1; c_2 \in$; $c_1 \neq 0$: (9) turns to

$$\Phi^2(-\Theta^2\Theta_u^2 c_1^2 + c_1^2(\Theta_{uu}\Theta^3 - \Theta^2\Theta_u^2) + \Theta\Theta_{uu} + 2\Theta_u^2)$$

$$+\Phi(-\Theta\Theta_u^2 c_1^2 + c_1^2(\Theta^2\Theta_{uu} - \Theta\Theta_u^2) + \Theta_{uu}) + 2\Theta^2 c_1^2 + 2 = 0 \tag{9}$$

The fact that the coefficient of the term Φ^2 in (11) must vanish leads to the contradiction $\Theta^2 = -\frac{1}{c^2}$
Case A.3. $\Phi_{vv} \neq 0$; taking a partial derivates of (8)

$$\left(\Theta^2\Theta_u^2(\Phi_{vv}\Phi^3 - \Phi^2\Phi_v^2) + \Theta^2\Phi\Phi_{vv} + \Phi^2\Phi_v^2(\Theta_{uu}\Theta^3 - \Theta^2\Theta_u^2) + G^2\Theta\Theta_{uu}\right) + 2\left(\Phi^2\Theta_u^2 + \Theta^2\Phi_v^2 + 1\right)$$

$$+\Theta\Theta_u^2\left(\Phi^2\Phi_{vv} - \Phi\Phi_v^2\right) +_{vv} +\Phi\Phi_v^2\left(\Theta^2\Theta_{uu} - \Theta\Theta_u^2\right) + \Phi\Theta_{uu} = 0 \tag{10}$$

with respect to Θ and Φ, which

$$\Big(A_1(\Theta)B_1(\Phi) + A_2(\Theta)B_3(\Phi) + 2A_3(\Theta)B_3(\Phi)\Big)$$

$$= -\Big(A_4(\Theta)B_4(\Phi) + A_5(\Theta)B_5(\Phi) + A_6(\Theta)B_6(\Phi)A_7(\Theta)B_7(\Phi)\Big) \tag{11}$$

where $A_1(\Theta) = \frac{d}{du}(\Theta_u^2)$, $A_2(\Theta) = \frac{d}{d\Theta}(\Theta_{uu}\Theta - \Theta_u^2)$, $A_3(\Theta) = \frac{d}{du}\left(\frac{1}{\Theta^2}\right)$, $A_4(\Theta) = \frac{d}{du}\left(\frac{\Theta_u^2}{\Theta}\right)$,
$A_5(\Theta) = \frac{d}{du}\left(\frac{1}{\Theta}\right)$, $A_6(\Theta) = \frac{d}{du}\left(\Theta_{uu} - \frac{\Theta_u^2}{\Theta}\right)$, $A_7(\Theta) = \frac{d}{du}\left(\frac{\Theta_{uu}}{\Theta^2}\right)$, $B_1(\Theta) = \frac{d}{dv}(\Phi_{vv}\Phi - \Phi_v^2)$,
$B_2(\Theta) = \frac{d}{dv}(\Phi_v^2)$, $B_3(\Theta) = \frac{d}{dv}\left(\frac{1}{\Phi^2}\right)$, $B_4(\Theta) = \frac{d}{dv}\left(\Phi_{vv} - \frac{\Phi_v^2}{\Phi}\right)$, $B_5(\Theta) = \frac{d}{dv}\left(\frac{\Phi_{vv}}{\Phi^2}\right)$,
$B_6(\Theta) = \frac{d}{dv}\left(\frac{1}{\Phi}\right)$, $B_7(\Theta) = \frac{d}{dv}\left(\frac{\Phi_v^2}{\Phi}\right)$.

Nevertheless, due to $A_7 \neq 0$; (10) can be divided by the product A_7B_7 as

$$\Big(\frac{A_1(\Theta)}{A_7(\Theta)}\frac{B_1(\Phi)}{B_7(\Phi)} + \frac{A_2(\Theta)}{A_7(\Theta)}\frac{B_3(\Phi)}{B_7(\Phi)} + 2\frac{A_3(\Theta)}{A_7(\Theta)}\frac{B_3(\Phi)}{B_7(\Phi)}\Big)$$

$$+\Big(\frac{A_4(\Theta)}{A_7(\Theta)}\frac{B_4(\Phi)}{B_7(\Phi)} + \frac{A_5(\Theta)}{A_7(\Theta)}\frac{B_5(\Phi)}{B_7(\Phi)} + \frac{A_6(\Theta)}{A_7(\Theta)}\frac{B_6(\Phi)}{B_7(\Phi)}\Big) = 0 \tag{12}$$

where the terms $\frac{A_1(\Theta)}{A_7(\Theta)}$, $\frac{B_1(\Phi)}{B_7(\Phi)}$, $\frac{A_2(\Theta)}{A_7(\Theta)}$, $\frac{B_3(\Phi)}{B_7(\Phi)}$, $\frac{A_3(\Theta)}{A_7(\Theta)}$, $\frac{B_3(\Phi)}{B_7(\Phi)}$, $\frac{A_4(\Theta)}{A_7(\Theta)}$, $\frac{B_4(\Phi)}{B_7(\Phi)}$, $\frac{A_5(\Theta)}{A_7(\Theta)}$, $\frac{B_5(\Phi)}{B_7(\Phi)}$,
$\frac{A_6(\Theta)}{A_7(\Theta)}$, $\frac{B_6(\Phi)}{B_7(\Phi)}$ must be constant for every Θ and Φ.
Since,
$\frac{A_1(\Theta)}{A_7(\Theta)} = a_1$, $\frac{A_2(\Theta)}{A_7(\Theta)} = a_2$, $\frac{A_3(\Theta)}{A_7(\Theta)} = a_3$, $\frac{A_4(\Theta)}{A_7(\Theta)} = a_4$, $\frac{A_5(\Theta)}{A_7(\Theta)} = a_5$, $\frac{A_6(\Theta)}{A_7(\Theta)} = a_6$.
a_1; a_2; a_3;a$_4$; a_5; a_6; a_7; a_8; a_9; a_{10}; a_{11}; $a_{12} \in$ and defined by

$$\Theta_u^2 = a_1\left(\frac{\Theta_{uu}}{\Theta^2}\right) + a_7, \tag{13}$$

$$\Theta_{uu}\Theta - \Theta_u^2 = a_2\left(\frac{\Theta_{uu}}{\Theta^2}\right) + a_8, \tag{14}$$

$$\frac{1}{\Theta^2} = a_3\left(\frac{\Theta_{uu}}{\Theta^2}\right) + a_9, \tag{15}$$

$$\frac{\Theta_u^2}{\Theta} = a_4\left(\frac{\Theta_{uu}}{\Theta^2}\right) + a_{10}, \tag{16}$$

$$\frac{1}{\Theta} = a_5\left(\frac{\Theta_{uu}}{\Theta^2}\right) + a_{11}, \tag{17}$$

$$\Theta_{uu} - \frac{\Theta_u^2}{\Theta} = a_6 \left(\frac{\Theta_{uu}}{\Theta^2} \right) + a_{12}. \tag{18}$$

So, by (17) and (18) we have,
$\frac{1}{a_3}\left(\frac{1}{\Theta^2} - a_9\right) = \frac{1}{a_5}\left(\frac{1}{\Theta} - a_{11}\right)$, this leads to a contradiction.

4 Examples

Example 4.1 *(The coordinate plane $y = 0$). The parametrization becomes*

$$r(u, z) = (u, 0, z), \quad (u, z) \in U \subset \mathbf{R} \times (0, \infty).$$

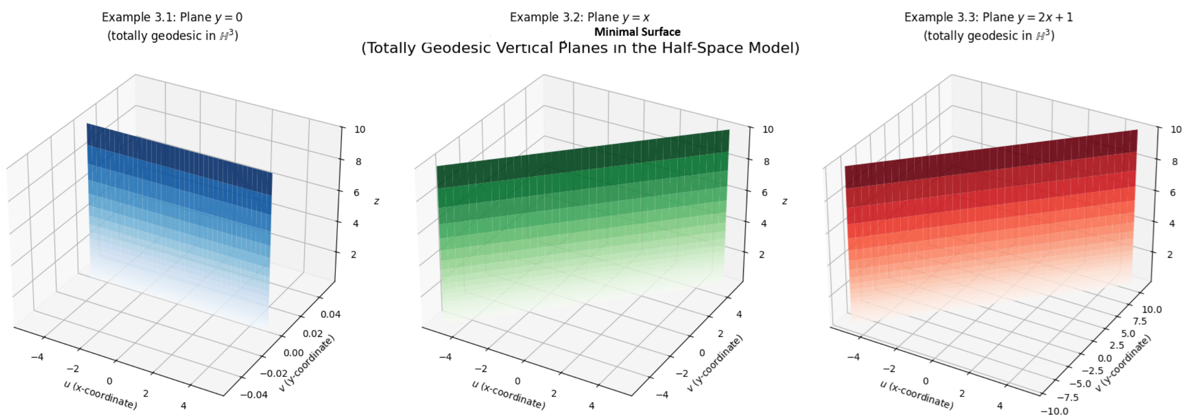
This describes the plane $y = 0$, $z > 0$, which is a vertical Euclidean plane perpendicular to the boundary $z = 0$. In $\mathbf{H}^3$, this is totally geodesic, with induced metric $ds^2 = \frac{du^2+dz^2}{z^2}$, mean curvature $H = 0$, and intrinsic Gaussian curvature -1.

Example 4.2 *(set $f(u) = u$, $c = 0$, and $g(z) = k$. The parametrization is*

$$r(u, z) = (u, u \cdot (k + 0) + 0 \cdot k, z) = (u, u, z).$$

This yields the surface $y = x$, $z > 0$, another vertical Euclidean plane. It is totally geodesic in $\mathbf{H}^3$, satisfying $H = 0$.

Example 4.3 *(A general linear vertical plane $y = mx+n$). Let $f(u) = mu+p$, $c = q$, and $g(z) = r$. For appropriate choices ($p+qr = n-mu\cdot q+\dots$, but reducing to linear), the surface becomes $y = mx+n$, $z > 0$, which is totally geodesic.*



5 Conclusion

This work has established a definitive rigidity theorem for minimal translation-factorable surfaces of type I in three-dimensional hyperbolic space, $\mathbf{H}^3$. Through a meticulous analysis of surfaces parameterized by $\mathbf{r}(u, v) = (u, v, \zeta(u)(\eta(v) + c) + c\eta(v))$ within the half-space model, we have proven that the vanishing of the mean curvature forces the surface to be a totally geodesic plane. The proof, hinging on a detailed computation of the fundamental forms and a systematic case analysis of the governing partial differential equation, demonstrates that any non-planar configuration leads to an unavoidable contradiction. This result underscores a profound geometric constraint inherent in spaces of constant negative curvature, starkly contrasting the rich variety of minimal translation surfaces admitted by Euclidean space. Our findings complete the classification program for this specific class of surfaces in $\mathbf{H}^3$ and suggest several promising avenues for future research.

Investigating translation-factorable surfaces of other types (e.g., where the roles of the coordinates are permuted or more complex functional decompositions are considered) within $\mathbf{H}^3$ and other homogeneous 3-manifolds.
Exploring the existence and classification of surfaces with constant non-zero mean curvature (CMC surfaces) within this family, which often present a richer theory than the minimal case.
Extending this rigidity analysis to the broader class of (ϵ)-surfaces in hyperbolic space, where the shape operator satisfies a specific polynomial equation, to see if similar constraints hold.
Examining the stability of the established minimal examples and the global geometric properties of the surfaces studied in this work.

This work thus not only closes a specific classification problem but also opens the door to a deeper exploration of how ambient curvature governs the existence and diversity of surfaces defined by simple geometric constructions.

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