The double sequences and applications to convergences of compound mean

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Abstract: In this paper, it is established that the square mean root is a hybrid mean, obtained relationship between square mean root and root mean square and justified that this hybrid mean is Schur concave and Schur geometric convex. As an application, it is proved that the double sequence of compound mean involving square mean root converges quickly.

1. Introduction

The concept of mathematical means was introduced and studied by Greek mathematicians during the fourth century A.D., particularly within the Pythagorean School, where it was rooted in the study of proportions and their significance [1]. Over time, numerous researchers have contributed to and advanced this field, motivated by its wide-ranging applications in various branches of science and technology.

Relationships between series and important means [2], exploring the connections between Greek means and functional means [3], and introducing and studying the Gnan mean for both two and *n* variables [4]. They also investigated homogeneous functions and, as an application, derived several inequalities involving means [5].

Furthermore, they studied the Oscillatory mean and Oscillatory-type means in the context of Greek means. Their work led to the proposal of several new means, their generalizations, and numerous related inequality results ([6], [7], [8], [9], [10], [11], [12]). A substantial body of work on Schur-convexity and properties of means has been documented in ([13],[14],[15],[16],[17],[18],[19]) while studies focused on the refinement of mean inequalities can be found in ([20], [21], [22]).

In the literature, it is well known that for any two real numbers a and b the Arithmetic mean $A(a\,,b) = \frac{a+b}{2}$, Geometric mean $G(a\,,b) = \sqrt{ab}$, Harmonic mean $H(a\,,b) = \frac{2ab}{a+b}$ and Contra Harmonic mean $C(a\,,b) = \frac{a^2+b^2}{a+b}$ represent fundamental types of means frequently studied in mathematical analysis.

2. Definitions and Lemmas

Few basic definitions and lemmas are listed which are associated to this study.

Definition 2.1:[1] A Mean is defined as a function $M: R_+^n \to R_+$ which satisfies the following properties:

$\min(a_1, a_2, a_3, \dots, a_n) \le M(a_1, a_2, a_3, \dots, a_n) \le \max(a_1, a_2, a_3, \dots, a_n), \forall\, a_i \ge 0,\ 1 \le i \le n,$

Definition 2.2:[1] For all real numbers a and b, the Power mean is defined as;

$$M_r(a\,,b) = \begin{cases} \left(\frac{a^r + b^2}{2}\right)^{\frac{1}{r}}, & r \ne 0 \\ \sqrt{ab}\,, & r = 0 \end{cases} \tag{1}$$

Remark: For $r = 1$, $M_1(a\,,b) = \frac{a+b}{2}$ is Arithmetic mean (AM).

For $r = 2$, $M_2(a\,,b) = \sqrt{\frac{a^2+b^2}{2}}$ is Root mean square (RMS).

Definition 2.3:[1] The double sequences in terms of G – Complementary to Square mean root and Square mean root is defined by:

$$a_{n+1} = {}^iSMR(a_n\,, b_n) = \frac{4\,a_n b_n}{(\sqrt{a_n} + \sqrt{b_n})^2} \text{ and } b_{n+1} = SMR(a_n\,, b_n) = \left(\frac{\sqrt{a_n} + \sqrt{b_n}}{2}\right)^2.$$

Definition 2.4:[1] The order C_n is said to be Log – Convex if $C_n^2 \le C_{n+1}\, C_{n-1}$, otherwise it is Log – Concave.

Definition 2.5:[1] The order "$(a_n)_{\,n\ge 0}$" and "$(b_n)_{\,n\ge 0}$" given $a_{n+1} = {}^iSMR(a_n, b_n)$ and $b_{n+1} = SMR(a_n\,, b_n)$, $n \ge 0$ is called a Gaussian double sequence.

Definition 2.6: [1] A mean N is called P – Complementary to M if it satisfies $P(M\,,N) = P$. Suppose a given mean M has a unique G – Complementary mean N is denoted by $N = M^{(G)} = \frac{G^2}{M}$. The G - Complementary mean is called Inverse.

Lemma 2.7:[1] For $\phi(x) = x^2$ and $a = (a_0,\ a_1,\ a_2)$ is just the determinant of Vander Monde's matrix of the 2^{nd} order takes the form.

$$V(a; r = 2, k = 0) = \begin{vmatrix} 1 & a_0 & a_0^2 \\ 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \end{vmatrix} = (a_1 - a_0)(a_2 - a_0)(a_2 - a_1).$$

Lemma 2.8:[1] Let $f(x)$ and $g(x)$ be two functions, then $f(x)$ is said to be convex with respect to $g(x)$ for $a \le b \le c$ if and only if $\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix} \ge 0.$

Lemma 2.9:[1] Let $\Omega \subseteq R^n$ be symmetric with non−empty interior geometrically convex set and let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω then φ is

$$i)\ (x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2} \right) \geq 0\ (\leq 0)$$

$$ii)\ (\ln x_1 - \ln x_2)\left(x_1 \frac{\partial\varphi}{\partial x_1} - x_2 \frac{\partial\varphi}{\partial x_2} \right) \geq 0\ (\leq 0)$$

$$iii)\ (x_1 - x_2)\left({x_1}^2 \frac{\partial\varphi}{\partial x_1} - {x_2}^2 \frac{\partial\varphi}{\partial x_2} \right) \geq 0\ (\leq 0)$$

is a Schur convex (concave), Schur−geometrically convex (concave) and Schur−Harmonically convex (concave) function respectively.

3. Square Mean Root

This section provides the definition and properties of square mean root.

Definition 3.1: For any real numbers a and b the square mean root is defined as:

$$SMR(a\,,b) = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2.$$

Remark 3.2: This is a particular case of power mean, $M_{\frac{1}{2}}(a\,,b) = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2$.

This is also a hybrid mean, $A(A\,,G) = \frac{1}{2}(A + G) = \frac{1}{2}\left(\frac{a+b}{2} + \sqrt{ab}\right) = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2$.

Definition 3.3: The inverse of square mean root is defined as:

$$^{i}SMR(a\,,b) = \frac{4ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}$$

4. Methodology

The analytical methods are applied in this article to prove the theorems by the use of partial derivatives and Taylor's series expansions. To justify the quicker convergence the graphs are obtained by the software, Origin.

5. Results and Discussions

This section provides the important results pertaining to square root mean.

Theorem 5.1: For $a < b$, verify that square mean root is a mean.

Proof: Consider, $a < b$; $\sqrt{a} < \sqrt{b}$ by adding $\sqrt{a}$ on both sides and divide by 2 gives,

$$\sqrt{a} < \frac{\sqrt{a}+\sqrt{b}}{2}.\ \text{Therefore,}\ a < \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2$$

Similarly, by adding $\sqrt{b}$ on both sides and divide by 2 gives,

$$\frac{\sqrt{a}+\sqrt{b}}{2} < \sqrt{b}.\ \text{Therefore,}\ \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 < b$$

Hence, square mean root is a mean.

Theorem 5.2: For $a < b$, the following inequality holds:
$H < G < SMR < A < C < RMS$.
Proof: This is to establish an inequality chain involving the square mean root (SMR).

$$SMR(a\,,b) - A(a\,,b) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2 - \left(\frac{a+b}{2}\right) = \frac{-2a-2b+4\sqrt{ab}}{4} < 0$$

$$SMR(a\,,b) - G(a\,,b) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2 - \sqrt{ab} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 > 0$$

$$SMR(a\,,b) - H(a\,,b) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2 - \frac{2ab}{a+b} = \frac{(a-b)^2 + 2\sqrt{ab}\,(a+b)}{2(a+b)} > 0$$

$$SMR(a\,,b) - C(a\,,b) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2 - \left(\frac{a^2+b^2}{a+b}\right) = \frac{-a^2 - b^2 + 2ab + 2\sqrt{ab}\,(a+b)}{2(a+b)} < 0$$

$$SMR(a\,,b) - RMS(a\,,b) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2 - \sqrt{\frac{a^2+b^2}{2}} = \frac{a+b+2\sqrt{ab}-2\sqrt{ab(a^2+b^2)}}{4} < 0$$

Combining the above, the following inequality chain is obtained.
$H < G < SMR < A < C < RMS$

Theorem 5.3: The square mean root with respect to arithmetic mean is convex, for all $a \le b \le c$.

Proof: Let, $SMR(a\,,b) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2$ and $A(a,b) = \frac{a+b}{2}$

If $a = a\,,\ b = 1$ then $f(a) = \left(\frac{\sqrt{a}\,+1}{2}\right)^2,\ g(a) = \frac{a+1}{2}$, then by lemma 2.8,

$$\Delta = \begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix} = \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \left(\frac{\sqrt{a}\ +\ 1}{2}\right)^2 & \frac{a+1}{2} \\ 0 & \left(\frac{\sqrt{b}\ +\ 1}{2}\right)^2 - \left(\frac{\sqrt{a}\ +\ 1}{2}\right)^2 & \left(\frac{b+1}{2}\right) - \left(\frac{a+1}{2}\right) \\ 0 & \left(\frac{\sqrt{c}\ +\ 1}{2}\right)^2 - \left(\frac{\sqrt{a}\ +\ 1}{2}\right)^2 & \left(\frac{c+1}{2}\right) - \left(\frac{a+1}{2}\right) \end{vmatrix}$$

$$= \frac{1}{4}\{c(\sqrt{b}-\sqrt{a}) + b(\sqrt{a}-\sqrt{c}) + a(\sqrt{c}-\sqrt{b})\}$$

Put $a = e^{-x}$, $b = e^{0x}$, $c = e^{x}$; $\Delta = \frac{1}{4}\left\{e^{x} - e^{-x} - \left(e^{\left(\frac{x}{2}\right)} - e^{\left(\frac{-x}{2}\right)}\right) - \left(e^{\left(\frac{x}{2}\right)} - e^{\left(\frac{-x}{2}\right)}\right)\right\}$

Apply the Taylor's series expansion leads to: $\Delta = \frac{x^3}{8} + \frac{x^5}{128} + \frac{x^7}{5120} + \cdots > 0$.
Hence the proof of the theorem.
The graph below represents the variation of AM versus SMR.

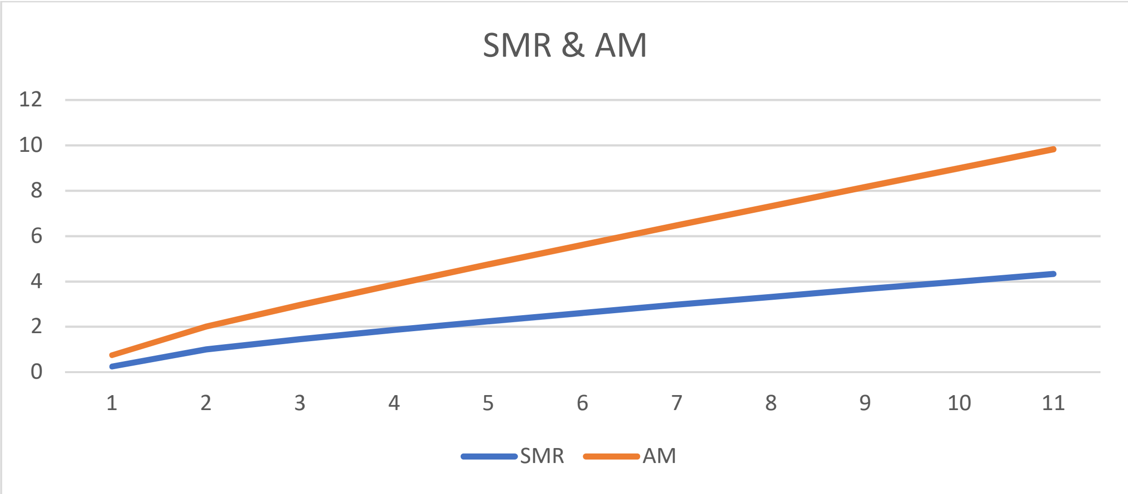

Figure 5.1: Variation of AM versus SMR

From the figure 5.1, the graph of AM lies above every chord (line segment) connecting any two points on the curve when those points are compared based on the corresponding values of SMR. Hence it is clear that SMR with respect to AM is convex.

Theorem 5.4: The square mean root with respect to geometric mean is concave, $\forall\, a \le b \le c$.

Proof: Consider, $SMR(a\,,b) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2$ and $G(a,b) = \sqrt{ab}$

If $a = a\,,\; b = 1$ then $f(a) = \left(\frac{\sqrt{a}\,+1}{2}\right)^2,\; g(a) = \sqrt{a}$ then by lemma 2.8,

$$\Delta = \begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix} = \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \left(\frac{\sqrt{a}\,+1}{2}\right)^2 & \sqrt{a} \\ 0 & \left(\frac{\sqrt{b}\,+1}{2}\right)^2 - \left(\frac{\sqrt{a}\,+1}{2}\right)^2 & (\sqrt{b})-(\sqrt{a}) \\ 0 & \left(\frac{\sqrt{c}\,+1}{2}\right)^2 - \left(\frac{\sqrt{a}\,+1}{2}\right)^2 & (\sqrt{c})-(\sqrt{a}) \end{vmatrix}$$

$$= \frac{1}{4}\{\sqrt{c}(b-a) + \sqrt{b}(a-c) + \sqrt{a}\,(c-b)\}$$

Put $a = e^{-x}$, $b = e^{0x}$, $c = e^{x}$

$$\Delta = \frac{1}{4}\left\{\left(e^{\left(\frac{x}{2}\right)} - e^{\left(\frac{-x}{2}\right)}\right) - (e^{x} - e^{-x}) + \left(e^{\left(\frac{x}{2}\right)} - e^{\left(\frac{-x}{2}\right)}\right)\right\}$$

Apply the Taylor's series expansion leads to: $\Delta = -\frac{x^3}{8} - \frac{x^5}{128} - \frac{x^7}{5120} - \cdots < 0$

Hence the proof of the theorem.

The graph below represents the variation of GM versus SMR.

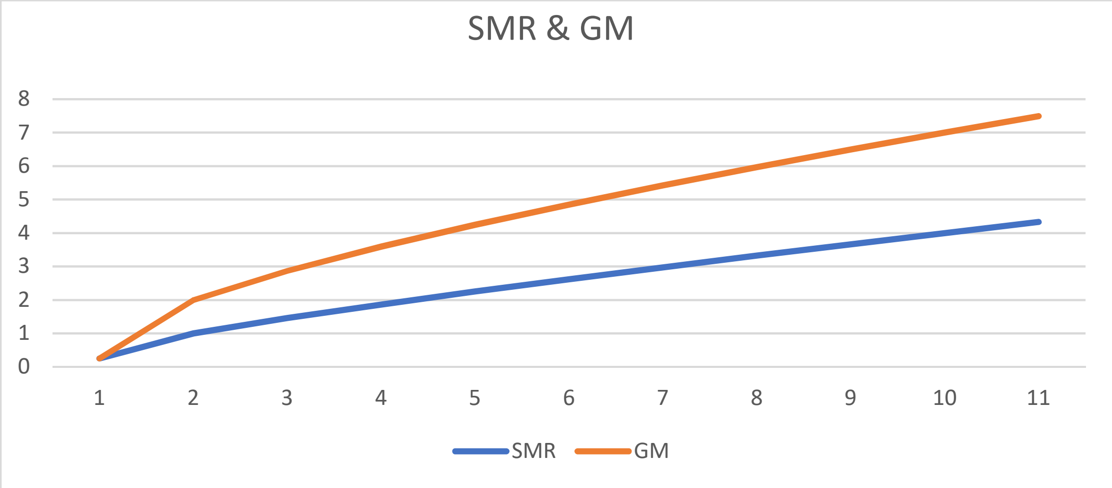

Figure 5.2: Variation of GM versus SMR

From the figure 5.2, the graph of GM lies above every chord (line segment) connecting any two points on the curve when those points are compared based on the corresponding values of SMR. Hence it is clear that SMR with respect to GM is convex.

Theorem 5.5: The square mean root with respect to contra harmonic mean is concave, $\forall\, a \le b \le c$.

Proof: Consider, $SMR(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2$ and $C(a,b) = \frac{a^2+b^2}{a+b}$

If $a = a,\ b = 1$ then $f(a) = \left(\frac{\sqrt{a}+1}{2}\right)^2,\ g(a) = \frac{a^2+1}{a+1}$ then by lemma 2.2,

$$\Delta = \begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix} = \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \left(\frac{\sqrt{a}+1}{2}\right)^2 & \frac{a^2+1}{a+1} \\ 0 & \left(\frac{\sqrt{b}+1}{2}\right)^2 - \left(\frac{\sqrt{a}+1}{2}\right)^2 & \left(\frac{b^2+1}{b+1}\right) - \left(\frac{a^2+1}{a+1}\right) \\ 0 & \left(\frac{\sqrt{c}+1}{2}\right)^2 - \left(\frac{\sqrt{a}+1}{2}\right)^2 & \left(\frac{c^2+1}{c+1}\right) - \left(\frac{a^2+1}{a+1}\right) \end{vmatrix}$$

$$\Delta = 1\left\{\left[\left(\frac{\sqrt{b}+1}{2}\right)^2 - \left(\frac{\sqrt{a}+1}{2}\right)^2\right]\left[\left(\frac{c^2+1}{c+1}\right) - \left(\frac{a^2+1}{a+1}\right)\right] - \left[\left(\frac{\sqrt{c}+1}{2}\right)^2 - \left(\frac{\sqrt{a}+1}{2}\right)^2\right]\left[\left(\frac{b^2+1}{b+1}\right) - \left(\frac{a^2+1}{a+1}\right)\right]\right\} < 0$$

Hence the proof of the theorem.

The graph below represents the variation of CHM versus SMR.

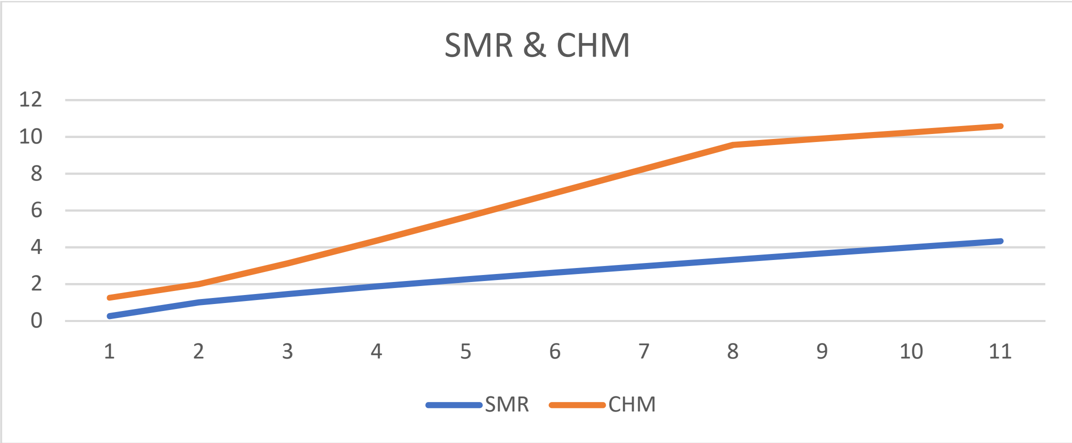

Figure 5.3: Variation of CHM versus SMR

From the figure 5.3, the graph of CHM lies above every chord (line segment) connecting any two points on the curve when those points are compared based on the corresponding values of SMR. Hence it is clear that SMR with respect to CHM is convex.

Theorem 5.6: For $a\,,b>0,$ the square mean root is Schur concave.

Proof: The square mean root is given by $SMR(a\,,b)=\left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2$

Differentiating partially with respect to a and b,

$$\frac{\partial(SMR)}{\partial a}=\frac{\sqrt{a}\,+\sqrt{b}}{4\sqrt{a}} \qquad \text{and} \qquad \frac{\partial(SMR)}{\partial b}=\frac{\sqrt{a}\,+\sqrt{b}}{4\sqrt{b}}$$

On subtracting, $\frac{\partial(SMR)}{\partial a}-\frac{\partial(SMR)}{\partial b}=\left(\frac{\sqrt{a}+\sqrt{b}}{4\sqrt{a}}\right)-\left(\frac{\sqrt{a}+\sqrt{b}}{4\sqrt{b}}\right)=\frac{b-a}{4\sqrt{ab}}$

$\therefore (a-b)\left(\frac{\partial(SMR)}{\partial a}-\frac{\partial(SMR)}{\partial b}\right)=(a-b)\left[\frac{b-a}{4\sqrt{ab}}\right]$

$=-\frac{(a-b)^2}{4\sqrt{ab}}<0,$ for all $a\,,b>0$. Hence the proof of the theorem.

Theorem 5.7: For $a\,,b>0\,,$ the square mean root is Schur geometric convex.

Proof: The square mean root is given by $SMR(a\,,b)=\left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2$

Differentiating partially with respect to a and b,

$$\frac{\partial(SMR)}{\partial a}=\frac{\sqrt{a}\,+\sqrt{b}}{4\sqrt{a}} \qquad \text{and} \qquad \frac{\partial(SMR)}{\partial b}=\frac{\sqrt{a}\,+\sqrt{b}}{4\sqrt{b}}$$

On subtracting, $\left(a\frac{\partial(SMR)}{\partial a}-b\frac{\partial(SMR)}{\partial b}\right)=a\left(\frac{\sqrt{a}+\sqrt{b}}{4\sqrt{a}}\right)-b\left(\frac{\sqrt{a}+\sqrt{b}}{4\sqrt{b}}\right)=\left(\frac{a-b}{4}\right)$

$\therefore (lna-lnb)\left(a\frac{\partial(SMR)}{\partial a}-b\frac{\partial(SMR)}{\partial b}\right)=(lna-lnb)\left(\frac{a-b}{4}\right)>0\,,$ for all $a\,,b>0$

Hence the proof of the theorem.

Theorem 5.8: For $a\,,b>0,$ the square mean root is Schur harmonic convex.

Proof: The Square Mean Root is given by $SMR(a\,,b)=\left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2$

Differentiating partially with respect to a and b,

$\frac{\partial(SMR)}{\partial a} = \frac{\sqrt{a}+\sqrt{b}}{4\sqrt{a}}$ and $\frac{\partial(SMR)}{\partial b} = \frac{\sqrt{a}+\sqrt{b}}{4\sqrt{b}}$

On subtracting, $\left(a^2\frac{\partial(SMR)}{\partial a} - b^2\frac{\partial(SMR)}{\partial b}\right) = a^2\left(\frac{\sqrt{a}+\sqrt{b}}{4\sqrt{a}}\right) - b^2\left(\frac{\sqrt{a}+\sqrt{b}}{4\sqrt{b}}\right)$

$$= \frac{(a-b)(a+b)+\sqrt{ab}\,(a-b)}{4}$$

$\therefore (a-b)\left(a^2\frac{\partial(SMR)}{\partial a} - b^2\frac{\partial(SMR)}{\partial b}\right) = \frac{(a-b)^2(a+b+\sqrt{ab})}{4} > 0$, for all , $b > 0$.

Hence the Square Mean Root is Schur Harmonic Convex.

Theorem 5.9: For $a < b$, verify that G-Complementary to square mean root is a mean.

Proof: Case (1): For $a < b$,Consider, ${}^{i}SMR(a\,,b) - a = \frac{4ab}{(\sqrt{a}+\sqrt{b})^2} - a = \frac{a\,(3b - a - 2\sqrt{ab})}{(\sqrt{a}+\sqrt{b})^2} > 0$

Which gives, $a < {}^{i}SMR(a\,,b)$ and hence Min $(a\,,b) = {}^{i}SMR(a\,,b)$.

Case (2): For $a <$ b, Consider, $b - {}^{i}SMR(a\,,b) = b - \frac{4ab}{(\sqrt{a}+\sqrt{b})^2} = \frac{b\,(b - 2\sqrt{ab} - 3a)}{(\sqrt{a}+\sqrt{b})^2} > 0$

Which gives, ${}^{i}SMR(a\,,b) < b$ and hence ${}^{i}SMR(a\,,b) <$ Max $(a\,,b)$.

Combining both the above cases, Min $(a\,,b) < {}^{i}SMR(a\,,b) <$ Max $(a\,,b)$.

Therefore, ${}^{i}SMR(a\,,b)$ is a Mean. Hence the proof of the theorem.

Theorem 5.10: The square mean root and G – Complementary to square mean root are related by the inequality $a < {}^{i}SMR(a\,,b) < SMR(a\,,b) < b$.

Proof: For $a < b$, consider $SMR(a\,,b) - {}^{i}SMR(a\,,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 - \frac{4ab}{(\sqrt{a}+\sqrt{b})^2}$

$$= \frac{(\sqrt{a}+\sqrt{b})^4 - 16ab}{2(\sqrt{a}+\sqrt{b})^2} > 0$$

Combining this inequality with lemma (4.1), gives $a < {}^{i}SMR(a\,,b) < SMR(a\,,b) < b$.
Hence the proof of the theorem.

Theorem 5.11: For two distinct positive real values $a_n < b_n$ the sequence $a_{n+1} = {}^{i}SMR(a_n\,,b_n)$ is monotonically increasing and the sequence $b_{n+1} = SMR(a_n\,,b_n)$ is monotonically decreasing. Satisfy,

min $(a\,,b) = a = a_0 < a_1 < \ldots < a_n < a_{n+1} < b_{n+1} < b_n \ldots < b_1 < b_0 = b =$ max $(a\,,b)$.

Proof: Let $a_{n+1} = {}^{i}SMR(a_n\,,b_n) = \frac{4\,a_n b_n}{(\sqrt{a_n}+\sqrt{b_n})^2}$ and $b_{n+1} = SMR(a_n\,,b_n) = \left(\frac{\sqrt{a_n}+\sqrt{b_n}}{2}\right)^2$

Consider, $\frac{a_{n+1}}{a_n} = \frac{4\,a_n a_n}{(\sqrt{a_n}+\sqrt{a_n})^2} > \frac{4\,a_n}{4\,a_n} = 1$, gives $a_{n+1} > a_n$ which holds for all n.

This proves that min $(a\,,b) = a = a_0 < a_1 < \ldots\ldots < a_n < a_{n+1} < \ldots\ldots$ (1)

Similarly, $\frac{b_{n+1}}{b_n} = \frac{(\sqrt{b_n}+\sqrt{b_n})^2}{4b_n} < \frac{4\,b_n}{4\,b_n} = 1$, gives $b_{n+1} < b_n$ which holds for all n.

This proves that $b_{n+1} < b_n < b_{n-1} \ldots\ldots < b_1 < b_0 = b =$ Max $(a\,,b)$ (2)

From (1) and (2), Hence the proof of the theorem.

Theorem 5.12: For n ≥ 0, $a_n < b_n$, the sequence $a_{n+1} = {}^{i}SMR(a_n\,,b_n)$ is log – concave and the sequence $b_{n+1} = SMR(a_n\,,b_n)$ is log – convex. If $a_n < b_n$ then $SMR(a_n\,,b_n)$ and

${}^{i}SMR(a_n, b_n)$ are given by $b_{n+1} = SMR(a_n, b_n) = \left(\frac{\sqrt{a_n}+\sqrt{b_n}}{2}\right)^2$ and $a_{n+1} = {}^{i}SMR(a_n, b_n) = \frac{4\,a_n b_n}{(\sqrt{a_n}+\sqrt{b_n})^2}$.

Proof: Consider, $\frac{a_n}{a_{n+1}} - \frac{a_{n-1}}{a_n} = \frac{(\sqrt{a_n}+\sqrt{b_n})^2}{4\,b_n} - \frac{(\sqrt{a_{n-1}}+\sqrt{b_{n-1}})^2}{4\,b_{n-1}}$

$$> \frac{1}{k}\left[\, a_n b_{n-1} + 2b_{n-1}(\sqrt{a_n b_n}\,) - b_n a_{n-1} - 2b_n(\sqrt{a_{n-1}\,b_{n-1}}\,)\right] > 0,$$

where k = $4b_n b_{n-1}$ [$\because$ $a_n > a_{n-1}$ and $-b_n > -b_{n-1}$]

So, $a_n^2 > a_{n+1}a_{n-1}$ and hence $a_{n+1} = {}^{i}SMR(a_n, b_n)$ is log – concave.

Consider, $\frac{b_n}{b_{n+1}} - \frac{b_{n-1}}{b_n} = \frac{4b_n}{(\sqrt{a_n}+\sqrt{b_n})^2} - \frac{4b_{n-1}}{(\sqrt{a_{n-1}}+\sqrt{b_{n-1}})^2}$

$$< \frac{4}{k}\left[\, b_n\left(\sqrt{a_{n-1}} + \sqrt{b_{n-1}}\right)^2 - b_{n-1}\left(\sqrt{a_n} + \sqrt{b_n}\right)^2\right] < 0,$$

where k = $\left(\sqrt{a_n} + \sqrt{b_n}\right)^2\left(\sqrt{a_{n-1}} + \sqrt{b_{n-1}}\right)^2$ [$\because$ $a_n > a_{n-1}$ and $b_n < b_{n-1}$]

So, $b_n^2 > b_{n+1}b_{n-1}$ and hence $b_{n+1} = SMR(a_n, b_n)$ is log – convex.

Hence the proof of the theorem.

Theorem 5.13: The Order "$(a_n)_{n\geq 0}$" and "$(b_n)_{n\geq 0}$" are defined in terms of G – Complementary to Square mean root ${}^{i}SMR(a_n, b_n)$ and Square mean root $SMR(a_n, b_n)$ which are Convergent to the common limit depicted as

$${}^{i}SMR(a_n, b_n) \otimes SMR(a_n, b_n) = G(a, b) = \sqrt{x}.$$

Proof: we have, $a_n < a_{n+1} < b_{n+1} < b_n$, $n \geq 0$:

$$b_{n+1} - a_{n+1} = \left(\frac{\sqrt{a_n}+\sqrt{b_n}}{2}\right)^2 - \frac{4\,a_n b_n}{(\sqrt{a_n}+\sqrt{b_n})^2}$$

$$b_{n+1} - a_{n+1} = \frac{(\sqrt{b_n}-\sqrt{a_n})^2\left[(\sqrt{a_n}+\sqrt{b_n})^2 + 4\sqrt{a_n b_n}\right]}{4\left(\sqrt{a_n}+\sqrt{b_n}\right)^2} < \frac{b_n - a_n}{4}$$

$b_n - a_n < \frac{b-a}{4}$ which tends to 0 as $n \to \infty$; Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

"$(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$" are monotonically increasing and monotonically decreasing sequences respectively.

Also, $a_{n+1}\, b_{n+1} = {}^{i}SMR(a_n, b_n) \times SMR(a_n, b_n) = \frac{4\,a_n b_n}{(\sqrt{a_n}+\sqrt{b_n})^2} \times \left(\frac{\sqrt{a_n}+\sqrt{b_n}}{2}\right)^2 = a_n b_n$

$\therefore a_{n+1}\, b_{n+1} = a_n b_n = a_{n-1} b_{n-1} =$ ------------- $= a_0\, b_0 = x$

Where ' x ' is a multiple of two positive real numbers; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$

The geometric mean of the sequences "$(a_n)_{n\geq 0}$ & $(b_n)_{n\geq 0}$" converges to a common limit $\sqrt{x}$. Hence the proof of the theorem.

6. VALIDATION OF QUICK CONVERGENCE OF NEW COMPOUND MEAN

In **[8],** authors discussed the method of extracting square root using Gaussian compound mean.

From the table 6.1, it is observed that the value of $\sqrt{2}$ correct to 6 decimal places converges in 4 iterations by Gaussian compound mean, but in case of the newly defined compound mean converges in 3 iterations. This justifies that the newly defined compound mean converges quickly.

Gaussian Compound Mean			New Compound Mean		
$H(a_n, b_n) \otimes A(a_n, b_n)$			$SMR(a_n, b_n) \otimes {}^{\wedge}iSMR(a_n, b_n)$		
n	a_n	b_n	n	a_n	b_n
0	1	2	0	1	2
1	1.333333	1.5	1	1.372583	1.457107
2	1.411765	1.416667	2	1.413898	1.414529
3	1.414211	1.414216	3	1.414214	1.414214
4	1.414214	1.414214	4	1.414214	1.414214

Table 6.1: The values of Gaussian compound mean and new compound mean

Graphical representation of the above data is as follows:

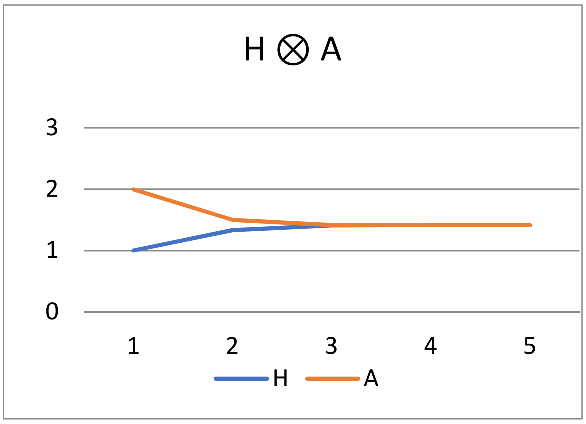

Figure 5.1(a): Gaussian compound mean

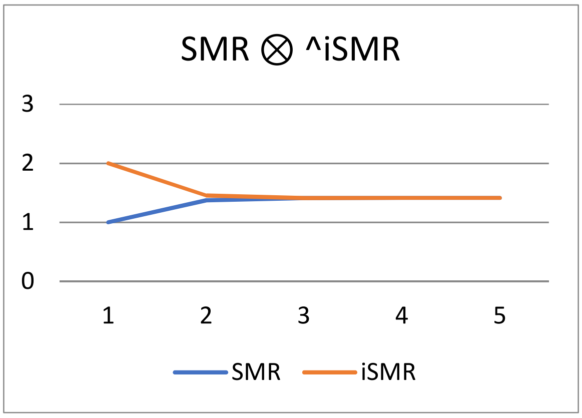

Figure 5.1(b): New compound mean

Figure 5.1(a) shows that the deviation of the Gaussian compound mean is negligible at iteration 3, while figure 5.1(b) shows that the deviation of the new compound mean is negligible at iteration 2 itself. This is the clear evidence that the convergence is quicker in case of new compound mean.

Combined graphical representation of the above data is as follows:

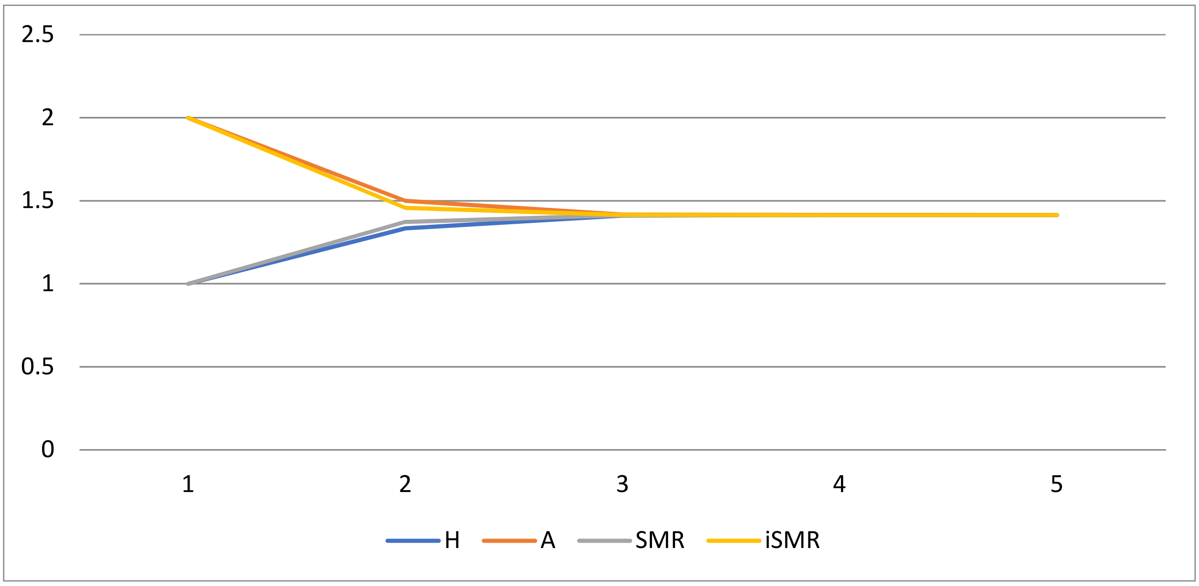

Figure 5.2: Comparison of Gaussian Compound Mean and New Compound Mean

In figure 5.2, the outer lines represent the deviation of the original Gaussian compound mean, while the inner lines indicate the new compound mean, which shows faster convergence.

7. Conclusion

In this article, initially proved that the square mean root and its invariant are means. It is established an inequality chain along with well-known means. Also, obtained few Schur convexity results. Finally, it is proved that the new compound mean is more advanced than the conventional Gaussian compound mean in computing surds.

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