FEKETE-SZEGÖ INEQUALITIES FOR NEW CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH (p, q)-DERIVATIVE OPERATOR

SARITHA G.P., SHILPA N. AND RENUKA DEVI K.

ABSTRACT. In this article we use (p, q)-derivative operator to introduce and study subclasses of analytic functions. Further, we derive Fekete-Szegö inequalities for the functions belonging to these new subclasses. Some special cases of the established results are also illustrated.

KEYWORDS AND PHRASES. Analytic functions, Subordination, (p, q)-derivative operator, Fekete-Szegö inequalities.
2010 AMS SUBJECT CLASSIFICATION: 30C45,30C50.

1. INTRODUCTION

Let $\mathcal{A}$ denote the family of normalized functions of the form

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{\zeta : \zeta \in \mathbb{C} \ and \ |\zeta| < 1\}$ and gratify the normalization conditions $f(0) = 0$ and $f'(0) = 1$.
A function f in $\mathcal{A}$ is said to be univalent in $\mathbb{U}$. We denote by $\mathbb{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.
Let f and g be two analytic functions in $\mathbb{U}$ then function g is said to be subordinate to f if there exists an analytic function ω in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|\omega(z)| < 1$ such that

$$g(\zeta) = f(w(\zeta)), \ (\zeta \in \mathbb{U}). \tag{1.2}$$

We denote this subordination by $g \prec f$.
In particular, if f is univalent in $\mathbb{U}$. The above subordination is equivalent to

$$f(0) = g(0) \ and \ f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

The theory of q-calculus plays an important role in many fields of mathematical, physical and engineering sciences. It has many applications in the field of special functions and other areas. The first application of the q-calculus was introduced by Jackson [13], [14]. Recently, a new generalization has been presented for the q-derivatives denoted by (p, q)-calculus. One may refer to [1, 3], [4], [5, 7, 15, 19, 21]. Let us recall some basic notations of (p, q)-calculus. For $0 < q < p \leq 1$ the $(p, q)-$derivative operator for the function f of the form (1.1) is defined by

$$D_{p,q}f(\zeta) = \begin{cases} \frac{f(p\zeta)-f(q\zeta)}{\zeta(p-q)}, \ \zeta \neq 0 \\ \\ f'(0), \ \zeta = 0. \end{cases} \tag{1.3}$$

From (1.3) we deduce that

$$D_{p,q}(f(\zeta) + g(\zeta)) = D_{p,q}f(\zeta) + D_{p,q}g(\zeta). \tag{1.4}$$

$$D_{p,q}(cf(\zeta)) = cD_{p,q}f(\zeta)$$

and the (p, q)-derivative of the function $h(\zeta) = \zeta^n$, is as follows

$$D_{p,q}h(\zeta) = [n]_{p,q}\zeta^{n-1} \tag{1.5}$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, p \neq q \tag{1.6}$$

which is a natural generalization of the q-number.

Clearly, we note that $[n]_{1,q} = [n]_q = \frac{1-q^n}{1-q}$ and $\lim_{q\to 1}[n]_{1,q} = n$.

$D_{p,q}h(\zeta) = h'(\zeta)$ as $p = 1$ and $q \to 1^-$, where $h'(\zeta)$ denotes the ordinary derivative of the function $h(\zeta)$ with respect to ζ.

The (p, q)-derivative operator of the function f, is defined as

$$\mathrm{D}_{p,q}f(\zeta) = 1 + \sum_{n=2}^{\infty}[n]_{p,q}a_n\zeta^{n-1}, \ \zeta \neq 0, \tag{1.7}$$

We now introduce the following classes of analytic functions by using principal of subordination and the (p, q)-derivative operator :

$$D^k_{\delta,\lambda,l,p,q}f(\zeta) = \zeta + \sum_{n=2}^{\infty} \left(\frac{(l + \delta - \lambda) + (1 - \delta + \lambda)[n]_{p,q}}{l + 1} \right)^k a_n\zeta^n \tag{1.8}$$

where $\delta, \lambda, l \geq 0$, $k \in \mathbb{N} \cup \{0\}$.

Let

$$D^k_{\delta,\lambda,l,p,q}f(\zeta) = \zeta + \sum_{n=2}^{\infty} L(\delta, \lambda, l, n)[n]_{p,q}a_n\zeta^n$$

where

$$L(\delta,\lambda,l,n)[n]_{p,q}=\left(\frac{(l+\delta-\lambda)+(1-\delta+\lambda)[n]_{p,q}}{l+1}\right)^{k}.$$

As $p=1$ and $q\to 1^-$, by specializing the parameters the new (p,q)-differential operator $D^k_{\delta,\lambda,l,p,q}$ reduces to various operators studied by Al-oboudi[6], Catas[8], Cho and Srivastava[9],Latha and Shilpa [16], Maslina Darus and Rabha W Ibrahim [18], Salagean [20] and Uralegaddi and Somanatha[22]. For example letting $q\to 1^-$, $\delta=\lambda$, $l=0$ we get Salagean operator and letting $q\to 1^-$, $\delta=1$, $l=0$ we get Al-oboudi operator.

Let P denote the class of all functions $\varphi(\zeta)$ which are analytic and univalent in $\mathbb{U}$ and for which $\varphi(\zeta)$ is convex with $\varphi(0)=1$ and $\Re\{\varphi(\zeta)\}>0$ for all $\zeta\in\mathbb{U}$.
By using principal of subordination and the (p,q)-derivative operator $D^k_{\delta,\lambda,l,p,q}$, we now introduce the following classes of analytic functions:

Definition 1.1. *A function $f(\zeta)$ belongs to the class $R^k_{\delta,\lambda,l,p,q}(\varphi)$ if it satisfies the following subordination condition*

$$D^n_{\delta,\lambda,l,p,q}f(\zeta)\prec\varphi(\zeta) \tag{1.9}$$

where $\varphi(\zeta)\in P$ and $0<q<p\le 1$.

Definition 1.2. *A function $f(\zeta)$ belongs to the class $N^k_{\delta,\lambda,l,p,q}(\varphi)$ if it satisfies the following subordination condition*

$$(1-\alpha)\frac{f(\zeta)}{\zeta}+\alpha D^k_{\delta,\lambda,l,p,q}f(\zeta)\prec\varphi(\zeta) \tag{1.10}$$

where $\varphi(\zeta)\in P$, $0\le\alpha\le 1$ and $0<q<p\le 1$.

In order to derive our main results, we need to following lemmas:

Lemma 1.3. *[12] If $p\in P$ then $|c_n|\le 2$ for each n, where P is the family of all functions p analytic in $\mathbb{U}$ for which*

$$\Re(p(\zeta))>0 \text{ and } p(\zeta)=1+c_1\zeta+c_2\zeta^2+\cdots$$

for $\zeta\in\mathbb{U}$.

Lemma 1.4. *[12] Let $p\in P$ be of the form $p(\zeta)=1+c_1\zeta+c_2\zeta^2+\cdots$. Then*

$$\left|c_2-\frac{c_1^2}{2}\right|\le 2-\frac{|c_1|^2}{2} \text{ and } |c_n|\le 2,\ n\in\mathbb{N}.$$

Lemma 1.5. *[17] If* $p(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \cdots$, $\zeta \in \mathbb{U}$ *is a function with positive complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leq 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the function given by

$$p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \text{ and } p(\zeta) = \frac{1+\zeta}{1-\zeta},\ \zeta \in \mathbb{U}.$$

The Fekete-Szegö problem is to find the coefficient estimates for the second and third coefficients of functions in any class of analytic functions having a specific geometric properties [11].In this paper, we obtain Fekete-Szegö inequalities for the classes $R^k_{\delta,\lambda,l,p,q}(\varphi)$ and $N^k_{\delta,\lambda,l,p,q}(\varphi)$.

2. Main Results

Theorem 2.1. *Let* $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots \in P$ *with* $\mathrm{B}_1 \neq 0$. *If* $f(\zeta)$ *belongs to the class* $R^k_{\delta,\lambda,l,p,q}(\varphi)$ *then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{L(\delta,\lambda,n)([3]_{p,q})[3]_{p,q}} \max\left\{1 - \left|\frac{B_2}{B_1} - \frac{L(\delta,\lambda,n)([3]_{p,q})[3]_{p,q}\mu B_1}{L(\delta,\lambda,n)([2]_{p,q})[2]^2_{p,q}}\right|\right\} \tag{2.1}$$

where μ *is a complex number and* $0 < q < p \leq 1$. *The result is sharp.*

Proof. If $f(\zeta) \in R^k_{\delta,\lambda,l,p,q}(\phi)$, by Definition (1.1) there is a Schwarz function $\omega(\zeta)$ in $\mathbb{U}$ such that

$$D^k_{\delta,\lambda,l,p,q}(f(\zeta)) = \varphi(\omega(\zeta)). \tag{2.2}$$

Now we define the function

$$p(\zeta) = \frac{1+\omega(\zeta)}{1-\omega(\zeta)} = 1 + p_1\zeta + p_2\zeta^2 + \cdots. \tag{2.3}$$

Since $\omega(\zeta)$ is a Schwarz function, we have $\mathbb{R}\{p(\zeta)\} > 0$ and $p(0) = 1$. Let

$$g(\zeta) = D^k_{\delta,\lambda,l,p,q}(\zeta) = 1 + d_1\zeta + d_2\zeta^2 + \cdots. \tag{2.4}$$

By using equations (2.2), (2.3) and (2.4) we obtain

$$g(\zeta) = \varphi\left(\frac{p(\zeta)-1}{p(\zeta)+1}\right)$$

since

$$\frac{p(\zeta)-1}{p(\zeta)+1} = \frac{1}{2}\left(p_1\zeta + \left(p_2 - \frac{p_1^2}{2}\right)\zeta^2 + \left(p_3 + \frac{p_1^3}{4} - p_1p_2\right)\zeta^3 + \cdots\right)$$

which yields

$$\varphi\left(\frac{p(\zeta)-1}{p(\zeta)+1}\right)=1+\frac{1}{2}B_1p_1\zeta+\left(\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2\right)\zeta^2+\cdots. \tag{2.5}$$

By equations (2.4) and (2.5) we obtain

$$d_1=\frac{1}{2}B_1p_1$$

$$d_2=\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2.$$

A simple computation gives

$$D^n_{\delta,\lambda,l,p,q}(f(\zeta))=1+L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}a_2\zeta+L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}a_3\zeta^2+\cdots$$

Using (2.4), we get

$$d_1=L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}a_2$$

$$d_2=L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}a_3$$

comparing the coefficients of ζ ad ζ^2 and simplifying we get

$$a_2=\frac{B_1p_1}{2L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}}$$

and

$$a_3=\frac{B_1}{2L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}}\left(p_2-\frac{p_1^2}{2}\right)+\frac{B_2^2p_1^2}{4L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}}$$

hence

$$a_3-\mu a_2^2=\frac{B_1}{2L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}}(p_2-\gamma p_1^2)$$

where

$$\gamma=\frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}\mu B_1}{[L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}\alpha]^2}\right)$$

□

Hence by Lemma (1.5), the result follows.

Note that, for suitable choices of parameters in Theorem (2.1) we get the following Corollary derived in [2].

Corollary 2.2. *Let $\varphi(\zeta)=1+B_1\zeta+B_2\zeta^2+\cdots\in P$, with $B_1\neq 0$. If $f(\zeta)$ given by (1.1) belongs to the class $R_{(q)}(\varphi)$ and μ is a complex number, then*

$$|a_3-\mu a_2^2|\leq\frac{\mathrm{B}_1}{[3]_{p,q}}\max\left(1-\left|\frac{B_2}{B_1}-\frac{[3]_{p,q}\mu B_1}{[2]_{p,q}^2}\right|\right) \tag{2.6}$$

The result is sharp.

Similarly, we can obtain upper bound for the Fekete-Szegö inequalities for functions belonging to the class $N^{n}_{\delta,\lambda,l,p,q}(\varphi)$ as follows.

Theorem 2.3. *Let $\varphi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + \cdots \in P$. If $f(\zeta)$ is given by (1.1) belongs to the class $N^{k}_{\delta,\lambda,l,p,q}(\varphi)$ then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{[(1-\alpha) + L(\delta,\lambda,n)([3]_{p,q})[3]_{p,q}\alpha]} \max\left\{1 - \left|\frac{B_2}{B_1} - \frac{\mu B_1[(1-\alpha) + L(\delta,\lambda,n)([3]_{p,q})[3]_{p,q}\alpha]}{[(1-\alpha) + L(\delta,\lambda,n)([2]_{p,q})[2]_{p,q}\alpha]^2}\right|\right\} \tag{2.7}$$

where μ is a complex number and $0 < q < p \leq 1$. The result is sharp.

Proof. If $f(\zeta) \in N^{k}_{\delta,\lambda,l,p,q}(\phi)$, by Definition (1.1) there is a Schwarz function $\omega(\zeta)$ in $\mathbb{U}$ such that

$$(1-\alpha)\frac{f(\zeta)}{z} + \alpha D^{k}_{\delta,\lambda,l,p,q}(f(\zeta)) = \varphi(\omega(\zeta)). \tag{2.8}$$

Now we define the function

$$p(\zeta) = \frac{1+\omega(\zeta)}{1-\omega(\zeta)} = 1 + p_1\zeta + p_2\zeta^2 + \cdots. \tag{2.9}$$

Since $\omega(\zeta)$ is a Schwarz function, we have $\mathbb{R}\{p(\zeta)\} > 0$ and $p(0) = 1$. Let

$$g(\zeta) = (1-\alpha)\frac{f(\zeta)}{\zeta} + \alpha D^{k}_{\delta,\lambda,l,p,q}(f(\zeta)) = 1 + d_1\zeta + d_2\zeta^2 + \cdots. \tag{2.10}$$

By using equations (2.8), (2.9) and (2.10) we obtain

$$g(\zeta) = \varphi\left(\frac{p(\zeta)-1}{p(\zeta)+1}\right)$$

since

$$\frac{p(\zeta)-1}{p(\zeta)+1} = \frac{1}{2}\left(p_1\zeta + \left(p_2 - \frac{p_1^2}{2}\right)\zeta^2 + \left(p_3 + \frac{p_1^3}{4} - p_1p_2\right)\zeta^3 + \cdots\right)$$

which gives

$$\varphi\left(\frac{p(\zeta)-1}{p(\zeta)+1}\right) = 1 + \frac{1}{2}B_1p_1\zeta + \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right)\zeta^2 + \cdots. \tag{2.11}$$

Using equations (2.10) and (2.11) we obtain

$$d_1 = \frac{1}{2}B_1p_1$$

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2.$$

A computation gives

$$(1-\alpha)\frac{f(\zeta)}{\zeta}+\alpha D^{n}_{\delta,\lambda,l,p,q}(f(\zeta)) = \quad 1+[(1-\alpha)+L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}\alpha]a_2\zeta+[(1-\alpha) \\ +L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}\alpha]a_3\zeta^2+\cdots$$

Inequality (2.10), yields

$$d_1=[(1-\alpha)+L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}\alpha]a_2$$

$$d_2=[(1-\alpha)+L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}\alpha]a_3$$

now comparing the coefficients of ζ ad ζ^2 and simplifying we get

$$a_2=\frac{B_1p_1}{2[(1-\alpha)+L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}\alpha]}$$

and

$$a_3= \quad \frac{B_1}{2[(1-\alpha)+L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}\alpha]}\left(p_2-\frac{p_1^2}{2}\right) \\ +\frac{B_2^2p_1^2}{4[(1-\alpha)+L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}\alpha]}$$

hence

$$a_3-\mu a_2^2=\frac{B_1}{2[(1-\alpha)+L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}\alpha]}(p_2-\gamma p_1^2)$$

where

$$\gamma=\frac{1}{2}\left(1-\frac{B_2}{B_1}-\frac{\mu B_1[(1-\alpha)+L(\delta,\lambda,l,n)([3]_{p,q})[3]_{p,q}\alpha]}{[(1-\alpha)+L(\delta,\lambda,l,n)([2]_{p,q})[2]_{p,q}\alpha]^2}\right)$$

□

Hence by Lemma 1.5, the result follows.

Note that, suitable choices of parameters in Theorem (2.3) yields the following Corollary derived in [2].

Corollary 2.4. *Let $\varphi(\zeta)=1+B_1\zeta+B_2\zeta^2+\cdots\in P$, with $B_1\neq 0$. If $f(\zeta)$ given by (1) belongs to the class $N_{(q)}(\varphi)$ and μ is a complex number, then*

$$|a_3-\mu a_2^2|\leq\frac{\mathrm{B}_1}{[(1-\alpha)+[3]_{p,q}\alpha]}\max\left(1-\left|\frac{B_2}{B_1}-\frac{[(1-\alpha)+[3]_{p,q}\alpha]\mu B_1}{[(1-\alpha)+[2]^2_{p,q}]}\right|\right) \tag{2.12}$$

The result is sharp.

References

1. Acar T., Aral A. and Mohiuddine S. A., On Kantorovich modification of (p ,q) -Baskakov operators, Journal of Inequalities and Applications, 98(1) (2016), 1-14.
2. Aini Janteng, Andy Liew Pik Hern and Rashidah Omar, Fekete-Szeg Functional of classes of analytic functions involving the q-derivative operator, Applied Mathematical Sciences, 14(10) (2020), 481-488.
3. Altnkaya S. and Yaln S., Certain classes of bi univalent functions of complex order associated with quasi-subordinataion involving (p,q)-derivative operator, Kragujevac Journal of Mathematics, 44(4) (2020), 639-649.
4. Araci S., Duran U., Acikgoz M. and Srivastava H. M., A certain (p,q)-derivative operator and associated divided differences, Journal of Inequalities and Applications, (1) (2016), 301.
5. Al-Hawary T., Yousef F. and Frasin B. A., Subclasses of Analytic Functions of Complex Order Involving Jacksons (p, q)-derivative, (2018).
6. Al-Oboudi F. M., On univalent functions defined by a generalized salagean operator, International Journal of Mathematics and Mathematical Sciences, 27 (2004), 1429–1436.
7. Bukweli-Kyemba J. D. and Hounkonnou M. N., Quantum deformed algebras: coherent states and special functions, arXiv preprint arXiv:1301.0116, (2013).
8. Catas A., On certain classes of $p-$valent functions defined by multiplier transformations, Proceedings of the International Symposium on Geometric Function Theory and Applications, Istabul, Turkey, (2007).
9. Cho N. E. and Srivastava H. M., Argument estimates of certain analytic functions defined by a class of multiplier transformations, Mathematical and Computer Modelling, 37(1-2) (2003), 39-49.
10. Duren P. L., Univalent Functions, Grundlehren der Mathematischen Wissenschaften; Springer-Verlag, New York, 1983.
11. Fekete, M., Szego, G., Eine bemerkung uber ungerade schlichte funktionen, J. Lond. Math. Soc., 8 (1933), 85-89.
12. Goodman A. W., Univalent functions, Mariner, Tampa, FL (1983).
13. Jackson F. H., On $q-$definite integrals, Quart. J. Pure Appl. Math., 41 (1910), 193-203.
14. Jackson F. H., $q-$difference equations, Am. J.Math., 32, 4 (1910), 305-314.
15. Jagannathan R.and Rao K.S., Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, arXiv (2006).
16. Latha S.and Shilpa N., On generalized differential and integral operators, International Journal of pure and applied mathematics, 117(20) (2017), 179-186.
17. Minda W. Ma, D., A unified treatment of some special classes of univalent functions, Proceedings of the Conference on Complex Analysis, (1992).
18. Maslina Darus and Rabha Ibrahim W., On new subclasses of analytic functions involving generalized differential and integral operators, European Journal of Pure and Applied mathematics, 4(1) (2011), 59-66.

19. Sadjang P. N., On the fundamental theorem of $(p, q)-$calculus and some $(p, q)-$Taylor formulas, arXiv:1309.3934 [math.QA], (2013).

20. Salagean G. S., Subclasses of univalent functions, Lecture notes in Math. SpringerVerlag, Berlin 1013 (1983), 362-372.

21. Srivastava H.M.,Raza N., AbuJarad E.S.A., Srivastava G. and AbuJarad M.H., Fekete-Szeg inequality for classes of (p, q)-starlike and (p, q)-convex functions. Revista de la Real Academia de Ciencias Exactas Fsicas y Naturales Serie A Matemticas, 113(4) (2019), 3563-3584.

22. Uralegaddi B. A. and Somanatha C., Certain classes of univalent functions, Current Topics in Analytic Function Theory, (1992), 371-374.

[1] Department of Mathematics, JSS Science and Technology University, Sri Jayachamarajendra College of Engineering, Mysuru 570 006, INDIA

sarithapswamy@jssstuniv.in

[2] Department of Mathematics, JSS College of Arts and Science, Ooty Road, Mysuru 570 025, INDIA

drshilpamaths@gmail.com

[3] Department of Mathematics, Govt. First Grade College for Women, K R Nagar - 571602, INDIA.

renukk84@gmail.com