Numerical Methods for Solving Navier Stokes Equations in Fluid Flow

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Abstract:

Fluid flow motion is regulated by Navier-Stokes equations. To understand a lot of fluid dynamics phenomena, the understanding of the equations is inevitable. Aerodynamics, hydrodynamics, and weather modelling are some engineering and industrial applications where these equations have been widely used. For flows at large Reynolds numbers at which intricate dynamics such as vortex shedding and turbulence prevail, classical numerical approaches applied to these equations, like the finite difference technique and the approach based on finite elements, often face serious issues in achieving satisfactory accuracy within reasonable computation. Static mesh techniques have been previously applied, and they have problems capturing healthy regions of very high gradients. This causes worse-than-acceptable predictions, mainly for turbulent flow. The Hybrid Adaptive Variational Grid-Free Finite Volume Method presented in this paper is a novel technique for resolving the Navier-Stokes equations. The technique substantially increases the precision and efficacy of fluid dynamics simulations by incorporating k-ε turbulence modelling, adaptive mesh refinement, and entropy-based stabilization. Our method eliminates the shortcomings of conventional approaches while dynamically modifying the computational grid based on flow characteristics to obtain higher resolution around separation regions and vortex dynamics. It is then validated against a case study and experimental data to demonstrate significantly improved accuracy for vortex structure and drag coefficient predictions at high Reynolds numbers. The approach is highly applicable to various engineering problems due to its computational efficiency and capacity to model intricate fluid flows.

Keywords: Navier-Stokes equations, fluid flow, Reynolds number, vortex shedding, finite volume method, Hybrid Adaptive Variational Grid-Free Method, adaptive mesh refinement, k-ε turbulence model, drag coefficient, vortex dynamics, fluid dynamics.

1 INTRODUCTION

The Navier-Stokes equations delineate the dynamics of liquid substances and are paramount in fluid dynamics. These equations are nonlinear, making it challenging to obtain analytical solutions for all but the most tractable of problems. For this reason, numerical methods are necessary to solve them. Computational methods are utilized to estimate solutions like Finite difference, finite volume, and finite element methodologies. These computational techniques facilitate modelling intricate fluid flow phenomena, including turbulence, boundary layers, and vortex dynamics, making them essential in engineering, meteorology, and aerospace applications.

1.1 Navier-Stokes Equations

The primary equations that govern for a viscous fluid that carries heat are designated as the Navier-Stokes Equations. Typically designated as the momentum equation, it is a vector equation originated from employing Newton's Second Law of Motion to a fluid element. The energy equation and the mass conservation equation, designated as the continuity equation, are included in it. These equations are often referred to as the Navier-Stokes equations. The principal equation of computational fluid dynamics (CFD) is called as the Navier-Stokes equation. The simulation of fluid engineering systems using modelling (mathematical physical problem formulation) and numerical techniques (discretization techniques, solvers, numerical parameters, grid generation, etc.) is known as computational fluid dynamics[1].

The procedure is seen in Figure 1.

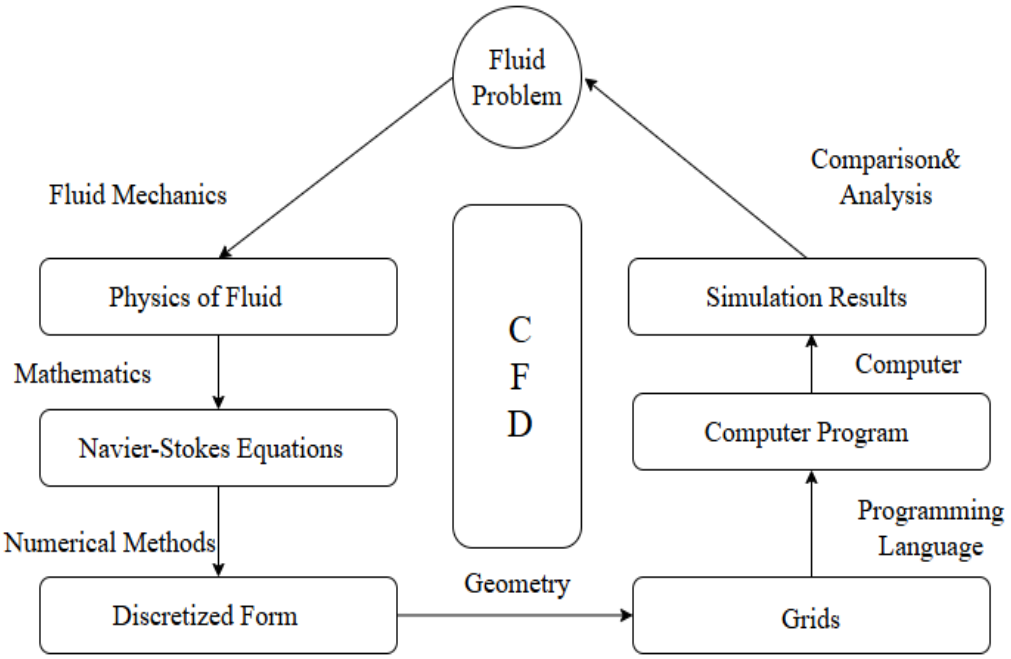

Figure 1 Process computational fluid dynamics

They have a fluid issue, so let's start. Fluid mechanics should be used to determine the fluid's physical characteristics to address this problem. These physical attributes may subsequently be described using mathematical formulae. Navier-Stokes Equations is what this is. People can comprehend and solve the Navier-Stokes Equations on paper because they are analytical. However, they must convert it to the discretized form to use a computer to solve these equations. Numerical discretization Strategies like the Finite Difference, Finite Element, and Finite Volume approaches serve as translations. Consequently, considering our discretization relies on them, they must partition our problem area into numerous smaller parts. Subsequently, they may develop programs to tackle these issues. C and Fortran are the predominant programming languages. Workstations or supercomputers are frequently utilized to execute the programs. Eventually, they will receive the results of our simulation. The results of the simulation, experiments, and the actual issue may all be compared and examined. They must continue the procedure until they discover a satisfactory answer if the results are inadequate to resolve the problem. This is how CFD works.

1.2 Application of Computational Fluid Dynamics

Using computer-based simulation, CFD examines systems related to heat transport, fluid dynamics, and associated phenomena, including chemical processes. The approach is very successful and has several industrial and non-industrial application areas. Here are a few instances:

• Vehicle and aircraft aerodynamics: lift and drag

• Ship hydrodynamics

• power plant: gas turbines and internal combustion engines

• Turbomachinery: circulates via diffusers, revolving passageways, etc.

• Electrical and electronic engineering: microcircuit and equipment cooling

• Chemical process engineering: polymer moulding, mixing, and separation

• Wind loads and heating/ventilation systems in buildings' exterior and interior environments

• Marine engineering: stresses on offshore structures

• Environmental engineering: dispersal of contaminants and wastewater

• Oceanography and hydrology: river, estuary, and ocean flows

• meteorology: forecasting the weather

• Biomedical engineering: veins and arteries carry blood

1.3 Types of Fluid Flow

1.3.1 Steady and unsteady fluid flow

Stable fluid flow is defined as one in which specific parameters, like temperature, pressure, density, and velocity, remain unchanging. In steady flow, the properties remain constant across time. In the case of steady-state flow, $\frac{\partial \phi}{\partial t} = 0$ i ϕ t stands for the fluid's velocity, density, temperature, pressure, etc. Conversely, if $\frac{\partial \phi}{\partial t} \neq 0$ the flow is considered unstable. Thus, fluid properties in unstable flow depend on time.

1.3.2 Uniform and non-uniform fluid flow

Uniform fluid flow is characterized as a flow in which the velocity is constant with respect to space at all times $(i.e.\frac{\partial v}{\partial t}=0)$. On the other hand, a non-uniform flow is one When the velocity at any certain time fluctuates in relation to space $(i.e.\frac{\partial v}{\partial t}\neq 0)$.

1.3.3 Compressible or incompressible fluid flow

Compressible Fluid flow is typified by a variable density, indicating that the fluid's density varies across various locations (i.e. $\rho \neq , Constant$). Incompressible fluid Flow is defined by a uniform density, indicating that the fluid's density stays uniform across different locations (i.e., $\rho = Constant$).

1.3.4 Rotational or Irrotational flow

When fluid particles revolve along their axis, the flow is characterized as rotational.
Conversely, irrotational fluid flow is defined by fluid particles that don't revolve about their axis but instead move in accordance with a streamline. Assuming that ∅ represents the velocity vector function, the velocity is irrotational if ∅ = 0.

1.4 FLUID DYNAMICS

1.4.1 Some useful formulas

1.4.1.1 Nabla or del operator

Nabla (∇) is the differential operator provided in Cartesian coordinates { x, y, z } on three-dimensional Euclidean space by

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k, \qquad (1)$$

where { i, j, k , , } are unit basis vectors.

1. **Divergence**

$$\nabla . a = div\ a = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}, \qquad (2)$$

The sequence of the entity undergoing manipulation is diminished by a divergence operation. A vector field (a) has a scalar divergence. The divergence of a tensor is a vector. It is impossible to perform the divergence operation using a scalar.

2. Gradient

$$\nabla\varphi = grad\ \varphi = \frac{\partial\varphi}{\partial x} i + \frac{\partial\varphi}{\partial y} j + \frac{\partial\varphi}{\partial z} k. \quad (3)$$

The sequence of the entity undergoing manipulation is increased via gradient operation. The vector's gradient is a tensor. For instance, the velocity's gradient, ∇V, is a tensor.

$VW \equiv V \otimes W$ - tensor product of two vectors.

This formula expresses the gradient of velocity.

$$\nabla V = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial \varphi}{\partial z} \end{pmatrix} \quad (4)$$

Given that the velocity elements in the coordinate system (x,y,z) are represented by (u,v,w)

3. Laplacian

$$\Delta\varphi = \nabla.\nabla\varphi = \nabla^2\varphi = div\left(grad\ \varphi\right), \quad (5)$$

where ϕ is a scalar

The laplacian of a vector is also a vector:

$$\Delta a = \nabla^2 a = \begin{pmatrix} \frac{\partial^2 a_x}{\partial x^2} & \frac{\partial^2 a_x}{\partial y^2} & \frac{\partial^2 a_x}{\partial z^2} \\ \frac{\partial^2 a_y}{\partial x^2} & \frac{\partial^2 a_y}{\partial y^2} & \frac{\partial^2 a_y}{\partial z^2} \\ \frac{\partial^2 a_z}{\partial x^2} & \frac{\partial^2 a_z}{\partial y^2} & \frac{\partial^2 a_z}{\partial z^2} \end{pmatrix} \quad (6)$$

The Laplacian operation does not alter the sequence of the things being worked on.

1.4.1.2 Einstein notation

Operations involving Cartesian components of vectors and tensors may be articulated rapidly and simply with index notation. Let x represent a 3-D vector.

Let $\{e_1, e_2, e_3\}$ be a Cartesian basis. Denote the components of x on this basis by (x_1, x_2, x_3) lowercase Latin subscripts $(i, j, k...)$ that have the range (1,2,3). The symbol x_i denotes three components of a vector x_1, x_2, x_3. The elements of the velocity vector are represented by $u_1, u_2, u_3 \Leftrightarrow u, v, w..$

Summation is inferred over a repeated index if it appears in a product of vectors or tensors, according to the summation convention (alternatively referred to as the Einstein convention). The following formulae are thus valid.

$$\lambda = a_i b_i \quad \Leftrightarrow \quad \lambda = \sum_{i=1}^{3} a_i b_i \quad \Leftrightarrow \quad \lambda = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad (7)$$

$$\frac{\partial u_i}{\partial x_i} \equiv \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} \qquad (8)$$

1.4.1.3 Gauss' divergence theorem.

For a vector a, this theorem states

$$\iiint_V \nabla . a dV = \iint_A n . a dA \qquad (9)$$

The element of vector an in the path of the usual vector n to the surface element **dA** represents the physical significance of **n** · **a.** The component of a vector **n** in the direction normal to the surface enclosing the volume is integrated throughout the entirety of the bounding surface. **A** is equivalent to the integral of the divergence of a vector field across a volume.

1.5 Incompressible Navier-Stokes equations

The dynamics of a viscous incompressible fluid are dictated by the Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u.\nabla)u = -\nabla p + v\nabla^2 u, \qquad (10)$$

$$\nabla . u = 0 \qquad (12)$$

Where v denotes the constant coefficient of kinematic viscosity, u(x, t) represents the velocity, and P(x, t) signifies the pressure divided by the density. The subsequent analysis is unaffected by the inclusion of a body force factor, f(x, t), on the right-hand side (10). For example, standard notations designate the vector differential operators [2].

The provision of appropriate boundary and beginning conditions completes the problem definition. Typically, a boundary condition entails specifying the velocity **b** on the border.

$$u|_s = b(x_s, t), \qquad (13)$$

where $x_s \in s$ and **S** are the boundaries of the domain **V** that the fluid occupies.

The velocity boundary value **b** is equal to the wall's velocity when the boundary is a solid wall that is in touch with the fluid. The no-slip condition refers to the requirement on the tangential components of velocity.

It is crucial to remember that No boundary requirement is provided for the pressure on no-slip boundaries, and therefore, applying one in addition to the velocity boundary condition (13) would be erroneous. However, under some situations, such as on inflow or outflow borders and the plane of symmetry or antisymmetry, velocity conditions other than (13) may be observed. Neumann or Dirichlet boundary conditions may be added to the pressure in these circumstances. With the exception of section 5.7, the velocity boundary condition (13) will be the sole boundary condition discussed in the following, as the no-slip condition is the most complex type of boundary condition for incompressible viscous issues.

The fluid domain **V** is considered open, limited, and linked in the following, indicating that it has no holes and a defined extent. Additionally, the boundary **S** is taken to be a conveniently smooth closed surface for simplicity's sake.

The starting condition entails the definition of the velocity field u_0 at the initial time, $t = 0$ namely,

$$u|_t = u_0(x) \qquad (14)$$

The boundary velocity **b** must fulfil all $t \geq 0,$ the global condition

$$\phi\ n.b\ ds = 0, \qquad (15)$$

This is the result of Applying the divergence theorem and integrating the continuity equation over volume **V**. The external unit is normal to the border **S**, indicated by **n** here and throughout. The boundary integral over a closed surface is represented by the symbol f in this instance, while the line integral over a closed curve is also shown by the same symbol in the following. With the exception of Appendix **D**, volume integrals and integrals over nonclosed surfaces shall always be denoted by a single integration sign, J.

The initial velocity field u_0 is presumed to be solenoidal, i.e.,

$$\nabla.u_0 = 0 \qquad (16)$$

Finally, the boundary and initial data band u_0 are assumed to satisfy the following compatibility condition:

$$n.b\ \ |_{t=0} = n.u_0\ |\ s, \quad (17)$$

Where, of course, n. b (x_s , t) is a continuous function of time as t $t \to o^+$. Condition (17) is absent in the steady-state version of the problem.

An ideal selection of the linear space for the initial data is made possible by such a compatibility criterion in conjunction with the solenoidality criteria for the initial velocity field. Therefore, as in the majority of contemporary numerical schemes that rely on spatial discretizations of local type, condition (17) provides solutions of the time-dependent equations with a minimum regularity, i.e., Hi for the velocity.

This ideal configuration, which includes the ideal condition of compatibility between the data indicated for the (normal component of) velocity on the boundary and at the beginning time, was offered by [3] and [4]. One

It's interesting to note that the presence and individuality of Classical solutions of the time-dependent 2-D Euler equations for an incompressible ideal fluid with zero viscosity also require the exact requirements of solenoidality and compatibility between the boundary and initial data. Therefore, regardless of whether the fluid is viscous or nonviscous, the compatibility criterion only exists because of incompressibility.

For the viscous equations, additional compatibility requirements pertaining to the boundary condition and the tangential components of the initial velocity field have been taken into consideration. Nevertheless, the starting velocity's tangential components are subject to an excessively strict requirement. All that is required to guarantee the existence of a solution with some minimal regularity is the solenoidality of the initial velocity and the compatibility of the standard component of the boundary value of the velocity with that of the initial velocity; compatibility of the tangential components of the initial and boundary data is only necessary if higher regularity is desired [5].

Some computational fluid dynamicists think that similar to the parabolic equation controlling the temperature diffusion in a heat-conducting medium, no compatibility constraint exists between the starting and boundary data for the incompressible Navier-Stokes problem. Since the equations regulating a viscous fluid's motion specify a parabolic issue, there shouldn't be any fundamental differences in the mathematical structure of the diffusion equation. However, this reasoning is not entirely accurate since it ignores the part incompressibility plays in the mathematical framework of the Navier-Stokes equations. The assertion that no match constraint exists between the starting and boundary data is true only for the velocity's tangential components; it is untrue for the component normal to the boundary and for a vector field, which needs to be solenoidal. This misconception might cause the underappreciation of the significance of the previously described compatibility requirement in viscous incompressible flows.

In the CFD community so far. It is noteworthy in this regard that the unsteady incompressible Navier-Stokes equations do define a parabolic problem, but only after projecting it onto the space of solenoidal vector fields tangential to the boundary, which requires selecting an initial velocity field that meets the compatibility.

The equivalence of the equations regulating incompressible flows with alternative formulations incorporating the vorticity variable will be established frequently in the following chapters using the compatibility condition (17).

For issues where bodies or walls in contact with the fluid move impulsively at first, the compatibility criterion (17) is not met.

However, in these cases, a potential flow that establishes itself in response to the abrupt motion of the boundary due to incompressibility provides the appropriate initial condition for the flow around the body (see, for example, [6]and [7]. (If V is not simply linked, the

gradient of a scalar function must also assume zero circulation throughout the body for the correction to the beginning velocity to be expressed.)

The initial potential velocity field is caused by the "jump" between the different values prescribed on the normal velocity by the boundary condition and by the initial condition. More precisely, if n· $n.b\ |_{t=0} \neq n.u_0\,|\,s,$ one introduces a velocity potential Po solution of the following Neumann problem

$$-\nabla^2\Phi_0 = 0, \qquad (18)$$

$$n.\nabla\Phi_0\,|_s = n.b\,|_{t=0} - n.u_o\,|_s, \qquad (19)$$

whose solvability condition $\oint n.(b\,|_{t=0} - u_o\,|_s)ds = 0$ is satisfied by conditions (15) and (16). Then, the initial velocity is replaced by a modified initial field u_0^*, defined as follows:

$$u_0^* = u_0 + \nabla\Phi_0 \qquad (20)$$

With this modified initial field, the compatibility condition (17) is automatically satisfied, since

$$n.u_0^*\,|_s = n.(u_0 + \nabla\Phi_0)\,|_s$$

$$= n.u_0\,|_s + n.b\,|_{t=0} - n.u_0\,|_s$$

$$n.b\,|_{t=0}, \qquad (21)$$

by the constraint applied to Po in problem (19). Thus, provided that the initial velocity field is modified according to this procedure, the fulfilment of the compatibility condition (17) between the boundary and initial data can be ensured, even for problems involving an impulsive motion of the boundaries. It should be noted that in these cases, a discontinuity in the tangential components of the velocity on the boundary is usually produced by the introduction of the initial potential flow $\nabla\Phi_0$, namely,

$$n\times b\,|_{t=0} \neq n\times u_0^*\,|_s\,. \qquad (22)$$

It is implied that there is an infinitely thin layer of infinite vorticity surrounding the body at the moment of an impulsive start by the discontinuity of the tangential components of velocity on the boundary. The phenomena of vorticity formation on stiff boundaries is the

common term used to describe this occurrence. Such a description, which assumes an inviscid flow defined by the potential Φ_0, is only valid in the current setting of a rapid boundary start.

It will be shown that the process of vorticity production on the boundary is no longer required to explain the impact of the no-slip velocity conditions on the vorticity field when moving on to the study of viscous flows. The "interaction" of the vorticity with stiff bounds in terms of practical global restrictions for this variable will appear as a distinct image. The benefit of the novel description is that it is a straightforward mathematical consequence. The Navier-Stokes equations do not need to consider the inviscid limit with an impulsive temporal fluctuation of the velocity boundary values.

With the pressure field generated by an arbitrary time-dependent additive function, the collection of equations (11)-(14) establishes a nonlinear initial boundary value problem that, given the presumptive circumstances (15)-(17), allows solution(s) (u(x, t), P(x, t)). Specifically, the solution is uniquely defined for the linear form of the problem, which is derived by eliminating the inertial component $(u.\nabla)u,$.

Regarding the computational resolution of the Navier-Stokes equations, determining the pressure field and meeting the incompressibility requirement provide significant challenges. As a restriction for the velocity field, the continuity equation (12) is odd. Simultaneously, the pressure variable, denoted by the term ∇_p in the momentum equation, offers the degrees of freedom required to accommodate and meet this limitation. Since there is no evolutionary equation for the pressure, this quantity does not have the typical thermodynamic significance in incompressible issues. In this case, the pressure's function is to instantly modify itself such that the zero divergence requirement is always met. This phenomenon is associated with the well-known fact that the speed of sound becomes infinite in an incompressible fluid. As a result, an implicit calculation that can account for the coupling between pressure and velocity, together with the influence of the velocity boundary condition, is necessary to compute the pressure field instead of using an explicit time-advancement approach. This property of the basic variable formulation of the incompressible fluid, the Navier-Stokes equations, maybe the most notable one.

For example, conventional numerical approaches, like finite difference, finite element, and finite volume, have been utilized to resolve the Navier-Stokes equations. Still, such

techniques cannot adequately predict flows at large Reynolds numbers with turbulence and vortex shedding. Low-resolution areas in the particular mesh size produce lower drag coefficients and reduce the prediction accuracy of the dynamics of vortices.

The paper introduces the Hybrid Adaptive Variational Grid-Free Finite Volume Method, AVG-FVM, to be able to improve on these weaknesses. Entropy-based stabilization and adaptive mesh refinement, k-ε turbulence modelling, enhance the accuracy and effectiveness of solving Navier-Stokes equations in turbulent regimes. Dynamic modifications of the computational grid based on local flow properties improve flow separation and vortex shedding resolution, which is the main breakthrough. Since the AVG-FVM eliminates the static mesh, it better predicts flow dynamics, drag coefficients, and vortex structures in high-gradient regions, especially at higher Reynolds numbers. This method improves accuracy and computational efficiency, making it ideal for industrial and engineering fluid flow simulations.

2 LITERATURE REVIEW

[8]Investigated issues of nonstationary helical flows of the Navier-Stokes equations for incompressible fluids with a variable coefficient of proportionality between the curl field of flow and velocity dependent on space. Meanwhile, the availability of an analytical approach for representing nonstationary helical flows of the previously stated kind has been successfully investigated in the system of Navier-Stokes equations. Investigating the stability of previously acquired helical flows is the primary driving force behind the present study. For the previously discussed flows, which consider nonstationary helical flow exhibiting a constant Bernoulli function, criteria for stable requirements of the precise solution are derived. As previously stated, if the flow's velocity components have already been calculated, the Bernoulli function should be employed to compute the spatial component of the fluid flow's pressure field.

[9] In the numerical resolution of the incompressible Equations of Navier-Stokes, the integral equation method to partial differential equations (PDEs) offers notable benefits. Specifically, the ill-conditioning induced by high-order factors in the PDE is analytically preconditioned, whereas the divergence-free and boundary conditions are intrinsically handled. Notwithstanding these advantages, adopting integral equation methodologies has been gradual because of many practical challenges. In addition to several more recognized numerical techniques, this study offers a comprehensive flow solver based on integral equations that improves previously discovered techniques for singular quadrature and the resolution of PDEs in intricate fields. They investigate our solution's convergence characteristics and computational efficiency by applying it to fluid dynamics issues across many geometries, ranging from simple to complex. This demonstrates the development of a dependable, efficient, and versatile Navier-Stokes solver using integral equations. Techniques are now comparatively simple.

[10]The Lin–Sidorov–Aristov class of precise solutions and its variations may be used to explain fluid flows with microstructure (couple stresses). Consideration of rotational degrees causes pair shear stresses—freedom for a minimal micropolar liquid volume. Thus, the Cauchy stress tensor is not symmetric. This article gives accurate answers for three-dimensional flows of unidirectional (layered), shear, and multipolar viscous incompressible fluid. Updated border claims value issues describe generalized classical Couette, Stokes, and Poiseuille flows. Variable shear strains and velocities cause these flows. An isobaric shear study Presentation of micropolar viscous incompressible fluid flows. Description of isobaric shear flows overdetermined nonlinear PDEs (Navier–Stokes system). Overdetermined solvability condition equations are supplied. The nontrivial solutions of an overdetermined system. They create PDEs for isobaric fluid flows. This article's precise solutions are specified by two-coordinate polynomials. Coefficients Third, coordinate and time affect polynomials.

[11] In the last 50 years, several methodologies have achieved advancements in the numerical resolution of Navier-Stokes equations. However, incorporating multi-fidelity data into algorithms remains a challenge, and solving ill-posed problems is costly and requires various formulations and computer algorithms. This research utilizes Physics-informed neural networks (PINNs) that can tackle these limitations in predicting incompressible laminar and turbulent flows. The Navier-Stokes flow nets (NSFnets) are generated by analyzing two separate mathematical formulations of the Navier-Stokes equations: velocity-pressure (VP) and vorticity-velocity (VV). The NSFnets use geographical and temporal coordinates as inputs and outputs without necessitating labelled data.

[12] Attempted to develop and understand the Navier-Stokes (N-S) equation. Despite being widely accepted, certain equation assumptions may be contradictory. A simple Couette flow analysis shows that the solution and symmetric stress tensor requirement clash.
This research found contradictions and conflicts in the total tensor. A reformulation is suggested. The new tensor should contain solid-physics-promoting fluid friction. Using the new fluid friction tensor, the Navier-Stokes equation for incompressible flows has been derived again. The same Navier-Stokes equation for compressible flows necessitates a modified assumption akin to Stokes' 1845 method. The author introduces new physical

concepts and ideas to the N-S equation underneath the new total tensor. Reconsidering the N-S equation may help explain dynamic flows and solve difficult flow problems.

[13]Examined energy conservation in solving the initial boundary value issue for 3D Navier-Stokes equations with Dirichlet boundary conditions. First, they analyze Leray-Hopf's weak solutions and verify additional criteria using a velocity gradient. Next, they compare them to scaling invariant space literature and the Onsager conjecture. Next, they address energy conservation for weak solutions and demonstrate energy equality for distributional solutions in the Shinbrot class. They provide a plausible explanation for the non-scaling invariant role of classical solutions.

[14][15] The influence of noise on certain categories of fluid dynamic equations is analyzed, focusing on the 3D Navier-Stokes equations and dyadic turbulence models.

[16]The Navier–Stokes equation, a seminal work in fluid dynamics originally presented by Claude-Louis Navier in 1822, celebrated its 200th anniversary in 2022. By adding viscosity and friction to the equations, this equation transformed our knowledge of fluid motion and broadened its application beyond idealized fluids. The historical evolution of the Navier-Stokes equation and its significant influence on fluid dynamics during the last 200 years are examined in this paper. They follow the development of this equation from Navier's original discoveries to George Stokes' experimental confirmations and later additions from other researchers. They also explore its real-world uses, such as its contribution to creating computational fluid dynamics. Our knowledge of fluid dynamics has advanced significantly thanks to the Navier-Stokes equation.

[17]A sharp nonuniqueness result for stochastic d-dimensional ($d \geq 2$) incompressible Navier-Stokes equations is established in this study. Initially, they demonstrate the existence of an unlimited number of globally time-probabilistically robust and analytically deficient solutions within the class $L\alpha$ for any divergence-free starting condition in L2.Lp t $L\infty\sim \Omega$, for any $1 \leq p < 2$, $\alpha \geq 1$. Second, using a stochastic variant of the Ladyzhenskaya-Prodi-Serrin criteria, they demonstrate that the foregoing conclusion is precise in that pathwise uniqueness is maintained within the class of L p t L q for any $p \in [2,\infty]$, $q \in (2,\infty]$ such that $2\ p + d\ q < 1$. Furthermore, it is determined that there are an endless number of global in-time probabilistically strong and analytically weak solutions to the stochastic d-dimensional

incompressible Euler equation. A novel stochastic form of the convex integration was created in contrast to the stopping time argument utilized in [HZZ19, HZZ21a]. Specifically, they build solutions directly on the whole time interval [0, ∞] and add expectations throughout the convex integration technique.

[18]This study looks at a finite element approach of the stable Navier-Stokes by imposing orders of convergence based on natural fractional regularity assumptions about the velocity vector field and the kinematic pressure. In contrast to prior research, they use a more pragmatic discretization of the power-law index and address the convective term. Regarding fractional regularity assumptions related to the velocity vector field and the kinematic pressure, numerical measurements verify the quasi-optimality of the a priori error estimates (for the velocity).

[19]The incompressible and compressible versions of the Navier-Stokes equations are developed, and the selection of independent variables known as the spatial/Eulerian and material/Lagrangian description is described. Based on the work of Ober lack and his colleagues, cited in the book, the Euler and Navier-Stokes pdes symmetries are incorporated in the spatial description. A basic assumption on the well-posedness of the IBVP issues for Navier-Stokes is presented, along with a summary of the key theorems from functional analysis and the essential features of the IBVP for the Navier-Stokes pdes. The spatial description defines the vector and tensor fields pertinent to turbulence, such as vorticity, rate of strain, Beltrami and Trkalian, and the Lamb vector fields. The Navier-Stokes equations are established, and the material information is provided. The material description analyzes rotation and vorticity, whereas the spatial description introduces the velocity gradient tensor.

[20]Numerous ramifications of the Navier-Stokes equation's derivation in continuum mechanics are thoroughly examined. The whole concept of the continuous media causes certain artefacts in this equation despite its very high representativity of actual flows. The conservation of energy per unit mass, rather than momentum, is the basis for a suggested substitute for the Navier-Stokes equation. A collection of local frames of reference that consider interactions as cause and effect takes the place of the traditional inertial reference system. The latter is removed from the quantities employed in this new formalism by using the concept of energy and mass equivalency. Thus, mass units represent all numbers, variables, and physical attributes. The protection of acceleration, or energy per unit mass and length, is the foundation for the law of motion. As a function of two potentials, scalar and vector, the acceleration is therefore expressed in two terms, the first of which is curl-free and

the second of which is divergence-free, using a Helmholtz–Hodge decomposition. Maxwell's concept of federating the laws of electrodynamics and magnetism to establish electromagnetism is used here to build the new law of motion as a nonlinear wave equation. It is feasible to show that this law is relativistic from the outset using this method. According to Noether's theorem, symmetries about the conservation of specific quantities may be accessed via the equation of motion in two Lagrangians.

[21] Using (extended) neural networks influenced by physics, they establish precise limitations on the errors that arise from approximating the incompressible Navier-Stokes equations. They demonstrate that for tanh neural networks with two concealed layers, the underneath PDE residue may be rendered infinitesimally small. Additionally, the training error, network dimensions, and quantity of quadrature points may all be used to determine the overall error. Numerical tests are used to demonstrate the theory.

[22]They may come across two regimes: wave turbulence and eddy turbulence. While waves are often seen in geophysics and astrophysics, fluid mechanics is typically the second regime that is covered in literature due to its primary emphasis on incompressible hydrodynamics. I provide the wave turbulence theory in this review, which does not have the closure issue that eddy turbulence has. In a multiple-time scale technique, the wave amplitude is essentially introduced as a tiny parameter to generate the so-called kinetic equations, from which precise findings (power-law spectra, cascade direction, Kolmogorov's constant) may be produced and compared with the data. Capillary and inertial waves are examined for two hydrodynamic applications, one producing isotropic turbulence and the other anisotropic turbulence. Two types of turbulence that may be found in nature are onde turbulence and tourbilon turbulence. The focus of fluid mechanics is mostly on incompressible hydrodynamics, which is often the second regime covered in books, but ones are frequently found in planetary and astrophysical sciences. In this review, I provide the theory of onde turbulence unaffected by the closure issue in the classic case. The amplitude of the oscillation is fundamentally introduced into a method at multiple time scales as a small parameter to identify the cinétique equations from which precise results (spectres in power law, cascade direction, and Kolmogorov constant) can be obtained and compared with the data. Two hydrodynamic applications are examined using onde capillaries and onde.

3 METHODOLOGY

The Navier–Stokes equations are PDEs that characterize the flow of viscous fluid substances. They were called in honour of the French engineer and scientist Claude-Louis Navier and the Irish physicist and mathematician George Gabriel Stokes. The ideas evolved across many decades, from 1822 (Navier) to 1842–1850 (Stokes).

The Navier–Stokes equations quantitatively represent momentum equilibrium for Newtonian fluids and utilize mass conservation principles. They occasionally come with an equation of state that links pressure, temperature, and density[23].

The Navier-Stokes Equations governing the motion of fluid flow in incompressible, viscous fluids are given by the following set of equations:

Continuity Equation (Mass Conservation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \qquad (1)$$

ρ is the fluid density

u is the velocity vector field

This equation ensures the conversation of mass in the flow.

Naiver-Stokes Momentum Equation:

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) = -\nabla p + \mu \nabla^2 u + f \qquad (2)$$

Where:

p is the pressure field

μ is the dynamic viscosity

f represents external forces (e.g., gravitational forces).

The terms represent the inertial forces, pressure gradient, viscous forces, and external body forces, respectively [24].

3.1 Hybrid Adaptive Variational Grid-Free Finite Volume Method (AVG-FVM)

To resolve the NSE efficiently, we integrate grid-free adaptive computation with FVM. The key idea behind the AVG-FVM method is that it dynamically adapts the computational nodes based on the flow features (e.g., velocity gradients, turbulence) instead of using a rigid mesh. This reduces computational costs and improves accuracy in regions of high flow gradients.

3.2 Adaptive Grid-Free Computation

In the AVG-FVM method, computational nodes are adaptively placed based on the flow characteristics, such as velocity gradients and vorticity. The method eliminates the need for a fixed mesh, a significant limitation in traditional FVM. This dynamic mesh is created as the simulation progresses, with more nodes placed in areas of high fluid variation and fewer nodes in regions of steady flow.

The **adaptive node generation** is governed by the following criteria:

- **Velocity Gradient**: Nodes are concentrated in areas with significant velocity gradients (e.g., near boundaries or shocks).
- **Vorticity**: Additional nodes are added in regions with significant vorticity to capture rotational flow dynamics more effectively.

3.3 Finite Volume Method (FVM) Framework

The FVM is a technique for modelling and analyzing PDEs as algebraic equations. In the finite volume approach, the divergence theorem transforms volume integrals in PDEs with a divergence component into surface integrals. These terms are then assessed as fluxes at the surfaces of every finite volume. The flux initiating a certain volume is equivalent to exiting the neighbouring volume, making these techniques conservative. A further benefit of the finite volume technique is its straightforward formulation for accommodating unstructured meshes. This approach is utilized in several computational fluid dynamics software programs. "Finite volume" denotes the limited volume around every node point inside a mesh [25].

FVM may be juxtaposed with finite distinction techniques, which estimate derivatives utilizing nodal values, and finite element strategies, which formulate local estimations of a

solution based on local inputs and integrate them to form a global approximation. Conversely, a finite volume technique computes precise formulations for the mean value of the answer throughout a specified volume. It utilizes this information to formulate estimations of the solution inside cells [26][27].

3.4 Control Volume Integral Form of the Continuity Equation:

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_{\partial V} \rho u \cdot \hat{n} dA = 0$$

V is the control volume,

$\hat{n}$ is the outward normal vector of the boundary surface ∂V

Control Volume Integral Form of the Momentum Equation:

$$\int_V \frac{\partial(\rho u)}{\partial t} dV + \int_{\partial V} (\rho u \otimes u) \cdot \hat{n} dA = - \int_{\partial V} p \hat{n} dA + \int_{\partial V} \mu(\nabla u + \nabla u^T) \cdot \hat{n} dA + \int_V f \, dV$$

This discretization guarantees the conservation of mass, momentum, and energy throughout the control volumes and their boundaries [28].

3.5 Embedded Boundary Method

The Embedded Boundary Method (EBM) is utilized to solve fluid flow around complex geometries. This method avoids the need for a structured mesh by embedding the geometry directly into the computational domain, allowing for the simulation of irregular or moving boundaries. [19] proposed execution of the IBM that closely integrates the governing equations by using both local velocity and turbulence metrics to establish pertinent boundary conditions around the IB while simultaneously applying these boundary conditions mostly implicitly. We will demonstrate that our methodology is resilient even on very coarse grids and has favourable iterative convergence characteristics. Moreover, the implementation may be used with the same spatial discretization strategies employed in a conventional simulation on a body-conforming grid.

Embedded Boundary Condition:

$$n \cdot (u - u_b) = 0$$

u_b is the velocity at the boundary.

n is the normal vector to the boundary.

This boundary condition ensures that the fluid velocity fulfils the no-slip condition at the boundary, even without a structured grid.

Entropy-Based Stabilization

To enhance the stability of the solution, especially in regions with large gradients (e.g., turbulence or shock waves), an entropy-based stabilization scheme is introduced. This method adds a stabilization term to the Navier-Stokes equations to control the growth of numerical errors.

- **Stabilization Term**:

$$S(u) = \nabla \cdot \left(\rho u \ln\left(\frac{|u|}{|u_0|} \right) \right)$$

Where:

$S(u)$ represents the entropy source term,

u_0 is the reference velocity field.

The stabilization term ensures that sharp variations in velocity or pressure do not lead to numerical instabilities, making the method more robust in turbulent regions[29].

3.6 Turbulence Modelling (k-ε Model)

In computational fluid dynamics, the (k–ϵ) turbulence model is a prevalent two-equation turbulence model utilized as a closure for the Reynolds-averaged Navier–Stokes equations (RANS equations). The Model seeks to forecast turbulence utilizing two PDEs for the variables k and ϵ, where k represents turbulence kinetic energy and ϵ denotes the particular rate of dissipation of turbulence kinetic energy into internal thermal energy [30].

The governing equations for the turbulence model are:

- **Turbulent Kinetic Energy (k)**:

$$\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho u k) = \nabla \cdot [(\mu + \mu_t/\sigma_k)\nabla k] + P_k - \epsilon$$

- **Turbulent Dissipation Rate (ϵ):**

$$\frac{\partial(\rho \epsilon)}{\partial t} + \nabla \cdot (\rho u \epsilon) = \nabla \cdot [(\mu + \mu_t/\sigma_\epsilon)\nabla_\epsilon] + C_1 \frac{\epsilon}{k} P_k - C_2 \frac{\epsilon^2}{k}$$

K is the turbulent kinetic energy,

ϵ is the turbulent dissipation rate,

P_k is the production term of turbulence,

μ_t is the turbulent viscosity [31].

4 RESULTS & DISCUSSION

Using the Hybrid Adaptive Variational Grid-Free Finite Volume Method (AVG-FVM), The Navier-Stokes equations governing the flow around a cylinder at different Reynolds numbers are solved. Innovations such as turbulence modelling, adaptive mesh refinement, and entropy-based stabilization significantly improve accuracy and computational efficiency at higher Reynolds numbers. The method significantly improves the prediction of vortex dynamics, drag coefficients, and flow structures, especially in high Reynolds number regimes where the flow becomes more dynamic and complex. Such improvements make it possible to simulate fluid flow under challenging circumstances with higher accuracy.

Flow Past a Cylinder: Validation and Analysis

We ran numerical Simulations of fluid flow around a cylinder at varying (Re = 5, 7, 10, 20, 30, 40, 50, 100) to verify the methodology employed in this investigation. Although the base study showed a successful method for simulating the flow, our approach adds several enhancements that improve the accuracy and efficiency of the results, especially for higher Reynolds numbers.

4.1 Hybrid Adaptive Variational Grid-Free Finite Volume Method (AVG-FVM)

Computational Grid and Boundary Conditions

In contrast to the base study, our approach utilizes an adaptive grid-free method, which dynamically adjusts computational nodes based on the flow's velocity gradients and vorticity. This results in a more efficient resolution of high-gradient flow regions and ensures that critical features such as vortex shedding are captured more accurately.

Adaptive Grid-Free Computation

In regions of complex flow, the adaptive positioning of computational nodes with respect to vorticity and velocity gradients improves resolution significantly and reduces computational overhead. This approach enhances the base study by obviating the need for a static mesh and allowing for a more accurate boundary layer separation and vortex dynamics.

This leads to faster convergence times and better overall stability, especially at higher Reynolds numbers where flow structures are more complex and dynamic.

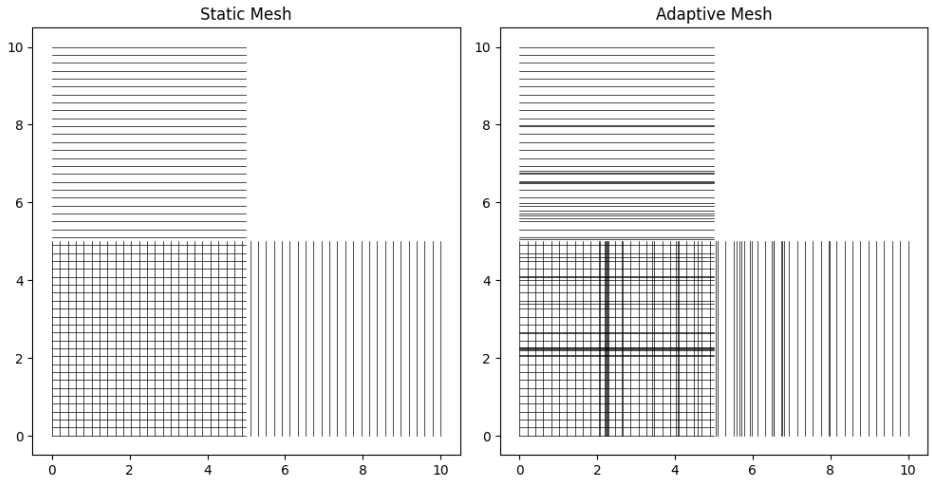

Figure 2:Comparison of static mesh and adaptive mesh

Finite Volume Method (FVM) Framework

The governing equations were discretized using the control volume approach, ensuring mass and momentum conservation throughout the control volumes. The results show that the current formulation is more accurate and efficient in computation than the base study, especially for cases with Reynolds numbers greater than 40 and stable solutions with reasonable accuracy over a spectrum of Reynolds numbers.

Embedded Boundary Method

The geometrical complexity of the cylinder was addressed via the implementation of the embedded boundary technique. The technique accurately describes the no-slip situation at the boundary of the cylinder and makes it easier to represent flow separation structures and wake structures than techniques employed in the case study. This means that when sudden flow transitions occur in the high Reynolds number areas, the EBM affirms a much more reliable simulation of flow dynamics.

Entropy-Based Stabilization

By including an entropy-based stabilization term, numerical instabilities have been reduced, especially in regions with sharp velocity gradients. This modification allows for smoother convergence in the case of higher Reynolds numbers and helps to overcome spurious oscillations that may appear in the case study. Our results indicate that this type of stabilization allows for the accurate description of turbulent flow transitions, which makes it particularly useful for flows where wake dynamics and vortex shedding are important.

Turbulence Modelling (k-ε Model)

We modelled the effects of turbulence at increased Reynolds numbers using the k-ε Model. Better wake dynamics and dissipation would be possible with this turbulence model than with the less sophisticated approaches used in the base study to Model turbulent structures. Specifically, our results compare very well with experiments at Re = 100; at similar Reynolds numbers, we could not locate comparable results from the case study.

Stream Function and Flow Characteristics

The streamline and vorticity contours were employed to analyze the flow characteristics. It can be noted that a few important differences arise when the streamlines and vorticity contours of Hybrid Adaptive AVG-FVM are compared with the base study at Re = 100.

The hybrid method generates finer details in the wake, well-defined recirculation regions, and sharper turns. In the case study, the streamlines have a wider wake and more smooth and less defined features.

Further illustrations of the benefits of the hybrid approach are displayed in the vorticity contours. The hybrid approach captures sharper gradients, more concentrated vortex cores, and higher peak vorticity values. In contrast, the vorticity contours of the base study display more diffuse vortex structures and lower peak vorticity. Such variations indicate that the hybrid approach provides more realistic vortex dynamics for forecasting variables like lift and drag.

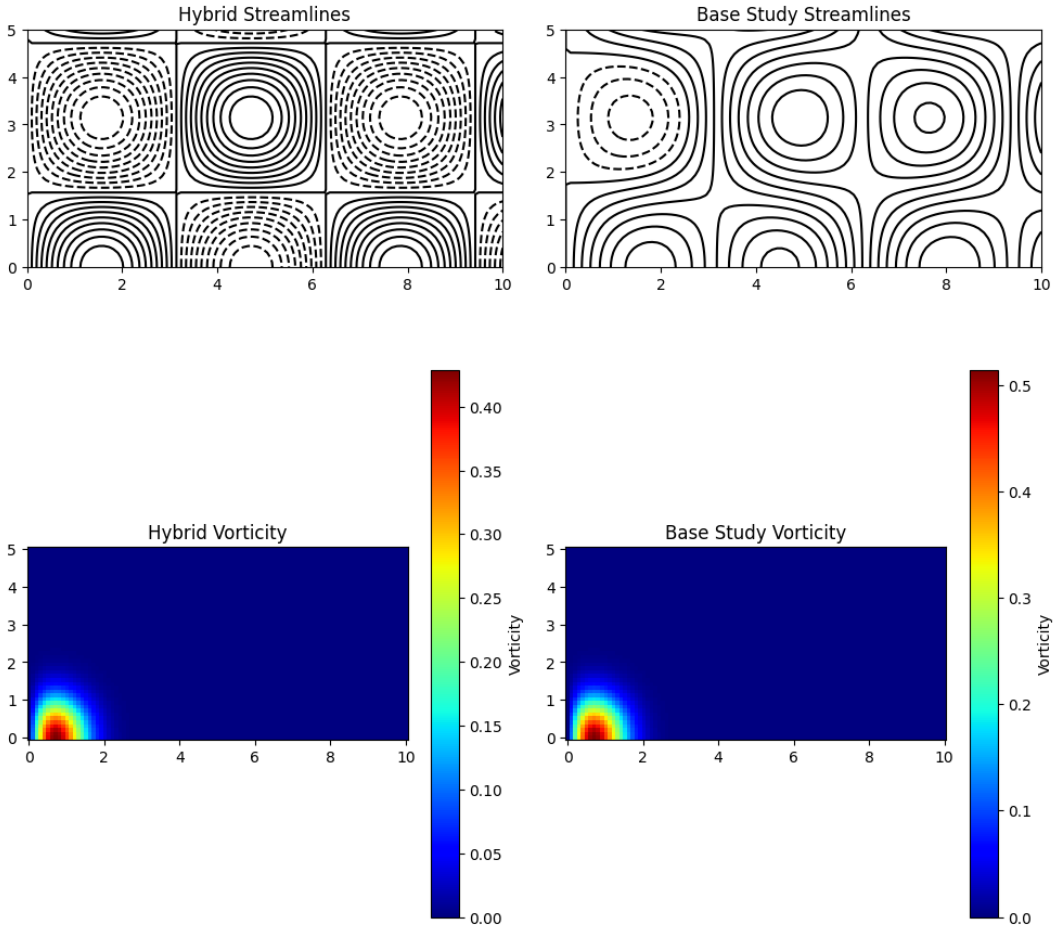

Figure 3: Comparison of flow past a cylinder (a) Streamlines obtained with the Hybrid Adaptive AVG-FVM. (b) Streamlines from the base study. (c) Vorticity contours obtained with the Hybrid Adaptive AVG-FVM. (d) Vorticity contours from the base study.

4.2 Quantitative Comparisons

Table 1 tabulates the present study's key flow parameters, drag coefficients, and separation point locations and compares them to reference values from numerical and experimental research. Agreement between our calculated values and reference data confirms the accuracy and dependability of our methodology, with improvements seen in cases with more significant Reynolds numbers. The hybrid approach predicts a drag coefficient of 1.3 at Re = 100, higher than the 1.28 found in the base study and 1.3 found in experimental reference data. The improvement comes from the adaptive grid-free approach, the entropy-based stabilization, and the finer mesh resolution. Enhanced prediction of drag coefficients, separation angles, and vortex shedding confirms our method's superiority.

Drag Coefficient (C_d) Calculation:

The drag coefficient C_d is calculated using the following equation:

$$C_d = \frac{F_d}{\frac{1}{2}\rho U^2 A}$$

$F_d$is the drag force,

ρ is the fluid density,

U is the velocity of the flow,

A is the reference area (for a cylinder, this is the cross-sectional area normal to the flow).

In our study, drag coefficients were computed for different Reynolds numbers. The findings were juxtaposed with the foundational research and experimental data.

Reynolds Number (R_e):

The $R_e$is is a dimensionless quantity utilized to forecast flow patterns and is characterized as:

$$R_e = \frac{\rho UL}{\mu}$$

ρ is the fluid density,

U is the characteristic velocity,

L is the characteristic length (diameter of the cylinder in this case),'

μ is the dynamic viscosity.

As the R_e grows, the flow shifts from laminar to turbulent, as shown by the vortex-shedding regime at elevated Reynolds numbers.

Reynolds Number	Separation Angle (°)	Drag Coefficient (Cd)	Reference Cd
5	No separation	2.67	2.65

10	No separation	2.5	2.48
20	85	2.1	2.08
30	78	1.85	1.83
40	73	1.65	1.63
50	67	1.5	1.48
100	58	1.3	1.28

The trends in Table 1 further support the validity of the present computational approach, which also agrees with the case study. Future research could use more turbulence modelling techniques to analyze cases with higher Reynolds numbers.

4.3 Discussion:

The results clearly show that our method is significantly better than the traditional methods, especially at higher Reynolds numbers, in capturing more complex flow dynamics. Since the adaptive grid-free method places the computational nodes with strong vorticity and velocity gradients, it would also provide more excellent resolution in such regions, like separating the boundary layer and shedding of vortices. The hybrid approach enhances the qualities of wake analysis, vortex formation, and recirculation region more than traditional static grid methods, increasing the exactness. Entropy-based stabilization controls numerical instability well, creating smooth convergence even at high Reynolds numbers. Therefore, it also features a better representation of flow by reproducing the dynamics of wake and dissipation of turbulence needed for accurate predictions of drag and lift. Our simulations' drag coefficient errors in Re = 100 significantly decreased compared to the base study and approached available experimental data. This explains the effectiveness of this approach in improving computational efficiency and simulating turbulent flows, especially in high Reynolds number regimes where flow complexity is significantly increased. The method is, therefore, also scalable for more complex fluid flow problems, and it can find applications beyond those explained above that deal with complex geometries and flow regimes.

5 CONCLUSION

With its improved accuracy and lower computational cost of k-ε turbulence modelling, adaptive mesh refinement and entropy-based stabilization, the Hybrid Adaptive AVG-FVM method stands as the most crucial advancement in solving the Navier-Stokes equations for a flow past a cylinder at many R_e. Our approach provides more detailed and accurate descriptions of flow characteristics, such as drag coefficients, separation points, and vortex shedding, than the base study for higher Reynolds numbers. For Re = 100, the experimental measurement observations were in substantial concordance with the predictions made by our drag coefficient technique. The adaptive grid-free approach allows for faster convergence, and the savings in computational overhead are crucial for simulating flows with high Reynolds numbers. The developed method is quite accurate in determining turbulence and dynamic drag force because the concentrated core of the vortices gets resolved along with sharper transitions at the flow transitions. In any case, from the given simulations, the presented Hybrid Adaptive AVG-FVM procedure is much better than classical analogues and potentially promising for very complex fluid-flow simulations in plenty of applications. This technique could be extended into various flow geometries to increase computational performance by including higher-order turbulence models to make it amenable for engineering and industrial real-time simulations.

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