Topological Index of a Cubic Heronian Mean Line Graphs

[1] Mahesh S A, [2] M Manjunatha, , [3] K.R. Karthik

[1] mahesh.s.a.gowda@gmail.com [2] mmanjunathapes@gmail.com

[3] karthikgowdru777@gmail.com

[1,2,3] Department of Mathematics, PET Research Foundation, P.E.S college of Engineering, Mandya-571401, India.

Abstract:

This paper examines the Weiner index of Cubic Heronian Mean Line Graphs (CHMLGs), which are a specific type of graph created by applying the Heronian mean to the edge weights of cubic graphs and then constructing their corresponding line graphs. The Weiner index is a topological index that quantifies the total of the shortest path distances between all pairs of vertices, serving as a tool to analyze the connectivity and compactness of these graphs. The study investigates how the Heronian mean influences the edge weights in cubic graphs and, in turn, affects the structure of their line graphs. The paper includes detailed calculations of the Weiner index for smaller CHMLGs and compares these findings with those of standard cubic graphs and their line graphs. Notable results indicate that incorporating the Heronian mean significantly changes the shortest path distances, leading to distinct behavior of the Weiner index for CHMLGs. These findings enhance our understanding of how non-linear edge weights impact graph properties, with possible applications in areas like network analysis, chemical graph theory, and molecular modeling.

1. Introduction:

In chemical graph theory, the Wiener index introduced by Harry Wiener, is a topological index of a molecule, defined as the sum of the lengths of the shortest paths between all pairs of vertices in the chemical graph representing the non hydrogen atoms inthe molecule. Wiener index can be used for the representation of computer networks and enhancing lattice hardware security. The Wiener index W(G) of a connected graph G is thesum of distances of all pairs of vertices of G : $W(G) = \sum_{(u,v)\subseteq V(G)} d(u, v)$

Average distance is one of the three most robust measures of network topology, along with its clustering coefficient and its degree distribution. Nowadays it has been frequently used in sociometry and the theory of social networks. Wiener index, defined asthe sum of distances between all (unordered) pairs of vertices in a graph, besides its crucialrole in the calculation of average distance, is the most famous topological index in mathematical chemistry. It is named after Wiener, who introduced it in 1947 for the purpose of determining boiling points of alkanes. Since then Wiener index has become oneof the most frequently used topological indices in chemistry, since molecules are usually modeled by undirected graphs. Other applications of this graph invariant can be found in crystallography, communication theory and facility location. Wiener index has also been studied in pure mathematics under various names: the gross status, the distance of a graph,the transmission of a graph etc. It seems that the first mathematical paper on Wiener indexwas published in 1976 [3]. Since then, a lot of mathematicians have studied this quantity very extensively. A great deal of knowledge on Wiener index is accumulated in survey papers. Nowadays it has been frequently used in sociometry and the theory of social networks.

Throughout this paper we consider only finite, undirected and simple graphs, that isgraphs without loops and multiple edges. Let G be a graph with vertex sets and V(G) and edge set E(G). The distance between the vertices u and edge of v is denotedby d(u,v) which is defined as the length of a shortest path between u and v in G. The degreeof a vertex u in G, which is written as deg(u), is the number of edges incident to u and theset of neighborhoods of v, denoted by $N_G(v)$ is the set of vertices adjacent to u.

A graph invariant is a real number related to a graph G which is invariant under graph isomorphism, that is it does not depend on the labeling or the pictorial representationof a graph. In chemistry, graph invariants are known as topological indices. Topological indices have many applications as tools for modeling chemical and other properties of molecules. The Wiener index is one of the most studied topological indices, both from a theoretical point of view and applications. This index was the first topological index to beused in chemistry. The Wiener index of a graph G, denoted by W(G), was introduced in 1947 by chemist Harold Wiener [6] as the sum of distances between all vertices of G.

2. Preliminary Concepts

Definition : 2.1

A line graph of a given graph G is a new graph $L(G)$ where the vertices of $L(G)$ correspond to the edges of G, and two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share a common vertex. Line graphs are important in the study of network properties because they help capture interactions between edges of the original graph.

Definition : 2.2

The **Weiner index** of a graph G is defined as the sum of the shortest path distances between all pairs of vertices: $W(G) = \sum_{1\leq i<j\leq p} d(i,j)$ where $d(i,j)$ represents the shortest path distance between vertices i and j, and p is the total number of vertices in the graph.

Calculation of the Weiner Index for CHMLGs : 2.3

The calculation of the **Weiner index** for a **Cubic Heronian Mean Line Graph** follows several steps:

Shortest Path Calculation: First, the shortest path distances between all pairs of vertices are calculated in the CHMLG. These distances depend on the edge weights, which are determined by the Heronian mean. The weight of each edge (e_1, e_2) in the line graph might be different from the simple case in a standard line graph, due to the use of the Heronian mean.

Distance Matrix: A matrix of distances between every pair of vertices is created. This distance matrix is crucial for computing the Weiner index.

Summing Distances: The Weiner index is then computed as the sum of the shortest path distances between all pairs of vertices in the line graph. This summation takes into account the Heronian mean-modified edge weights.

The modification introduced by the Heronian mean makes this calculation more complex than traditional cubic or line graph Weiner index calculations, as the distances between vertices in the line graph are influenced by the non-linear edge weights.

3. Main Outcomes

In this paper we consider some Line graphs, such as Path, Cycle, Comb and some corona of a graph. Our result on generalized for a finite set of graphs (see [7]). A Linear graph P_n is a walk in which all the nodes are distinct. A Polygon graph C_n is a Closed Path. The graph obtained by joining a single pendant edge to each vertex of a Path is called a Toothed graph or Comb.

Remark: 3.1 The line graph of a linear is also a linear.

Remark: 3.2 The line graph of Polygon is also Polygon.

Proposition : 3.3

The Weiner index of Comb $G = P_5 \odot K_1$ is $W(G) = n + (n-1) \times 4 + (2n-2) \times 3 + (2n+2) \times 2 + (2n+1) \times 1$

Proof:

Let G be a graph obtained from a Path $P_n = u_1 u_2 \dots \dots u_n$ by joining the vertex u_i to $v_i; 1 \leq i \leq n$. Graph G of Comb $P_5 \odot K_1$ is displayed below.

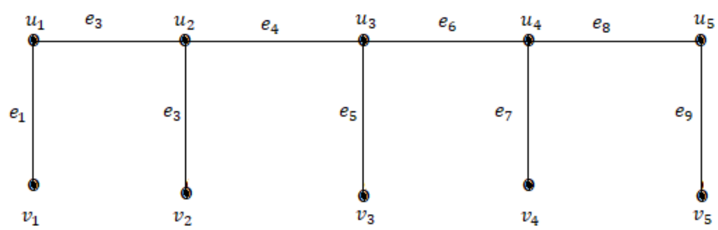

Figure : 1

Let e_i be the vertices of $L(G)$. The Line graph $L(G)$ Comb $P_5 \odot K_1$ is shown in figure : 2

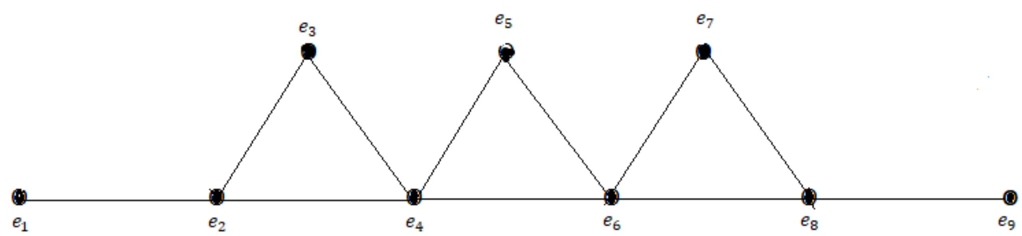

Figure : 2

In general, the Line graph $L(G)$ of Comb $P_n \odot K_1$ is shown in figure : 3

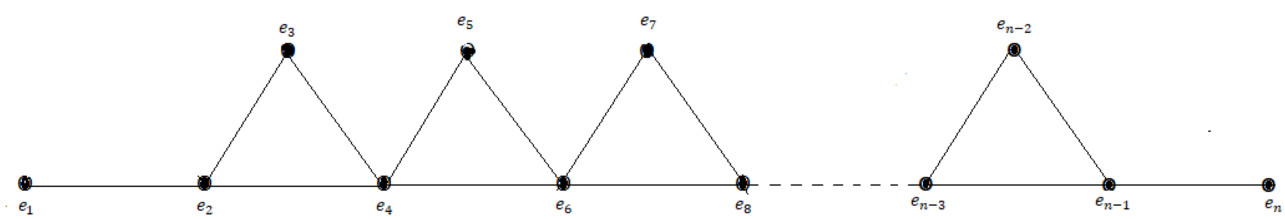

Figure : 3

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v)$$

$$W(P_5 \odot K_1) = \begin{gathered} d(u_1,u_2) + d(u_1,u_3) + d(u_1,u_4) + d(u_1,u_5) + d(u_1,u_6) + d(u_1,u_7) + d(u_1,u_8) + d(u_1,u_9) + \\ d(u_2,u_3) + d(u_2,u_4) + d(u_2,u_5) + d(u_2,u_6) + d(u_2,u_7) + d(u_2,u_8) + d(u_2,u_9) + \\ +d(u_3,u_4) + d(u_3,u_5) + d(u_3,u_6) + d(u_3,u_7) + d(u_3,u_8) + d(u_3,u_9) + \\ +d(u_4,u_5) + d(u_4,u_6) + d(u_4,u_7) + d(u_4,u_8) + d(u_4,u_9) + d(u_4,u_9) + \\ d(u_5,u_6) + d(u_5,u_7) + d(u_5,u_8) + d(u_5,u_9) + d(u_6,u_7) + d(u_6,u_8) + d(u_6,u_9) + \\ d(u_7,u_8) + d(u_7,u_9) + d(u_8,u_9) \end{gathered}$$

$$= \begin{gathered} (1+2+2+3+3+4+4+5) + (1+1+2+2+3+3+4) + \\ (1+2+2+3+3+4) + (1+1+2+2+3) + \\ (1+2+2+3) + (1+1+2) + \\ +(1+2) + (1) \end{gathered}$$

$$= (11 \times 1) + (12 \times 2) + (8 \times 3) + (4 \times 4) + (1 \times 5)$$

$$= n + (n-1) \times 4 + (2n-2) \times 3 + (2n+2) \times 2 + (2n+1) \times 1$$

$$\therefore \; W(P_5 \odot K_1) = n + (n-1) \times 4 + (2n-2) \times 3 + (2n+2) \times 2 + (2n+1) \times 1$$

Proposition : 3.4 The Weiner index of the graph $G = P_4 \odot K_{1,2}$ is $W(G) = (4n) + (4n+2) \times 1 + (5n+1) \times 2 + (3n+2) \times 3$

Proof:

Graph G of $P_4 \odot K_{1,2}$ is displayed below.

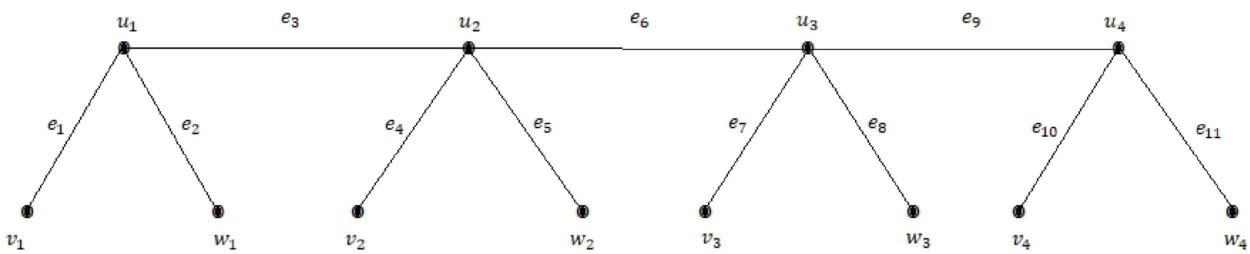

Figure : 4

The Line graph $L(G)$ of $P_4 \odot K_{1,2}$ is shown in figure : 5

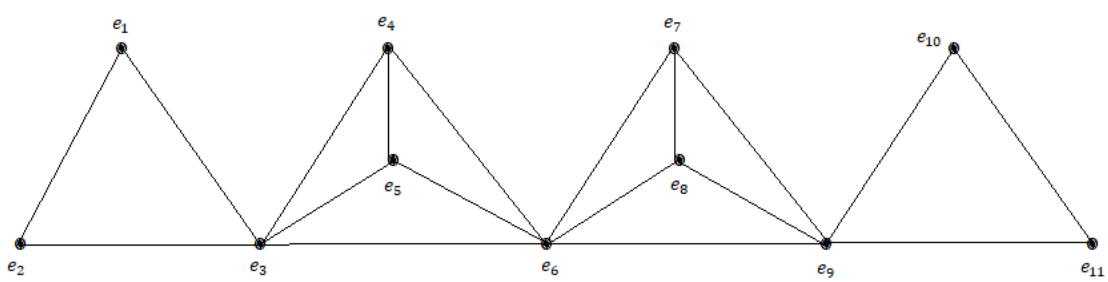

Figure: 5

In general, the Line graph of $P_4 \odot K_{1,2}$ is shown in figure: 6

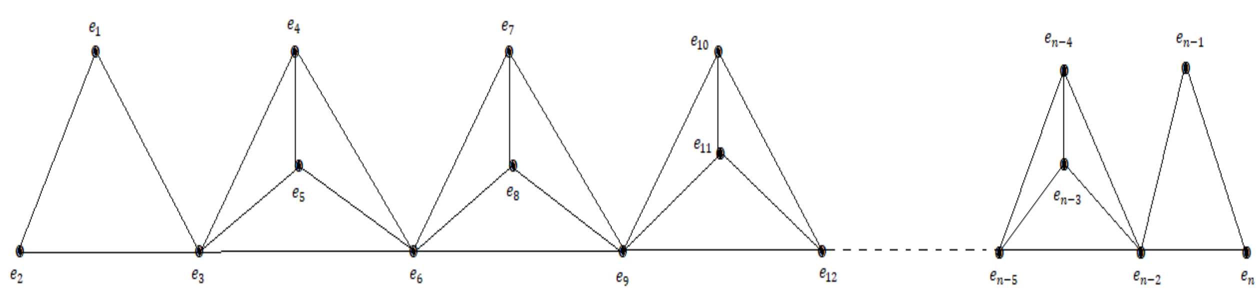

Figure: 6

$$W(G) = \sum_{u,v\in V(G)} d_G(u,v)$$

$$W\left(P_4 \odot K_{1,2}\right) = \begin{array}{c} d(u_1,u_2)+d(u_1,u_3)+d(u_1,u_4)+d(u_1,u_5)+d(u_1,u_6)+d(u_1,u_7)+d(u_1,u_8)+d(u_1,u_9)+ \\ d(u_1,u_{10})+ d(u_1,u_{11})+ d(u_2,u_3)+d(u_2,u_4)+d(u_2,u_5)+d(u_2,u_6)+ \\ d(u_2,u_7)+d(u_2,u_8)+d(u_2,u_9)+d(u_2,u_{10})+d(u_2,u_{11})+ \\ +d(u_3,u_4)+d(u_3,u_5)+d(u_3,u_6)+d(u_3,u_7)+d(u_3,u_8)+d(u_3,u_9)+d(u_3,u_{10})+d(u_3,u_{11}) \\ +d(u_4,u_5)+d(u_4,u_6)+d(u_4,u_7)+d(u_4,u_8)+d(u_4,u_9)+d(u_4,u_{10})+d(u_4,u_{11}) \\ d(u_5,u_6)+d(u_5,u_7)+d(u_5,u_8)+d(u_5,u_9)+d(u_5,u_{10})+d(u_5,u_{11})+d(u_6,u_7)+d(u_6,u_8)+ \\ +d(u_6,u_9)+d(u_6,u_{10})+d(u_6,u_{11})+d(u_7,u_8)+d(u_7,u_9)+d(u_7,u_{10})+d(u_7,u_{11})+d(u_8,u_9) \\ d(u_8,u_{10})+d(u_8,u_{11})+d(u_9,u_{10})+d(u_9,u_{11})+d(u_{10},u_{11}) \end{array}$$

$$= \begin{array}{c} (1+1+2+2+2+3+3+3+4+4)+(1+2+2+2+3+3+3+4+4)+ \\ (1+1+1+2+2+2+3+3)+(1+1+2+2+2+3+3)+ \\ (1+2+2+2+3+3)+(1+1+1+2+2)+ \\ +(1+1+2+2)+(1+2+2)+(1+1)+(1) \end{array}$$

$$= (18\times 1)+(21\times 2)+(12\times 3)+(4\times 4)$$

$$= (4n+2)\times 1+(5n+1)\times 2+(3n)\times 3+(n)\times 4$$

$$\therefore \;\; W(P_5 \odot K_1) = (4n)+(4n+2)\times 1+(5n+1)\times 2+(3n+2)\times 3$$

4. Weiner Index of Special Graphs

Proposition : 4.1

The Weiner index of Diamond graph is $W(G_d) = (n+1)\times 1+(n-1)\times 2$

Proof:

Let G_d be a Diamond graph with 4 vertices and 5 edges and $L(G_d)$ be the line graph of G_d with 5 nodes and 6 edges.

$$W(G) = \sum_{u,v\in V(G)} d_G(u,v)$$

$$W(G_d) = d(u_1,u_2)+d(u_1,u_3)+d(u_1,u_4)+d(u_2,u_3)+d(u_2,u_4)+d(u_3,u_4)$$

$$= (1+1+1)+(1+2)+1$$

$$= (5\times 1)+1$$

$$= (n+1)\times 1+(n-3)$$

$$\therefore \;\; W(G_d) = (n+1)\times 1+(n-3)$$

$$W(L(G_d)) = d(u_1,u_2)+d(u_1,u_3)+d(u_1,u_4)+d(u_1,u_5)+d(u_2,u_3)+d(u_2,u_4)+$$

$d(u_2,u_5)+d(u_3,u_4)+d(u_3,u_5)+d(u_4,u_5)$

$$=(1+2+2+1)+(1+2+2)+(1+1)+1$$

$$=(6\times 1)+(4\times 2)$$

$$=(n+1)\times 1+(n-1)\times 2$$

$$\therefore\ W(L(G_d))=(n+1)\times 1+(n-1)\times 2$$

Proposition : 4.2

The Weiner index of Bull graph is $W(G_b)=(n\times 1)+(n-4)\times 2$

Proof:

Let G_b be a Bull graph with 5 vertices and 5 edges and $L(G_b)$ be the line graph of G_b with 5 vertices and 7 edges.

$$\therefore W(G)=\sum_{u,v\in V(G)}\quad d_G(u,v)$$

$$W(G_b)=d(u_1,u_2)+d(u_1,u_3)+d(u_1,u_4)+d(u_1,u_5)+d(u_2,u_3)+d(u_2,u_4)+ d(u_3,u_4)+d(u_2,u_5)+d(u_3,u_4)+d(u_3,u_5)+d(u_4,u_5)$$

$$=(1+1+1+2+2)+(1+1+2)+(2+1)+3$$

$$=(5\times 1)+(4\times 2)+3$$

$$=(n\times 1)+(n-1)\times 2+(n-2)$$

$$\therefore\ W(G_b)=(n\times 1)+(n-1)\times 2+(n-2)$$

$$W(L(G_b))=\begin{matrix}d(u_1,u_2)+d(u_1,u_3)+d(u_1,u_4)+d(u_1,u_5)+d(u_2,u_3)+d(u_2,u_4)+\\ d(u_3,u_4)+d(u_2,u_5)+d(u_3,u_4)+d(u_3,u_5)+d(u_4,u_5)\end{matrix}$$

$$=(1+2+2+1)+(1+1+1)+(1+2)+1$$

$$=(7\times 1)+(3\times 2)$$

$$=(n\times 1)+(n-4)\times 2$$

$$\therefore\ W(L(G_b)_{=}(n\times 1)+(n-4)\times 2$$

Proposition : 4.3

The Weiner index of Fork graph is $W\left(G_f\right)=n\times 1+(n-2)\times 2$

Proof:

Let G_f be a Fork graph with 5 vertices and 4 edges and $L(G_f)$ be the line graph of G_f with 4 vertices and 4 edges.

$$\therefore W(G) = \sum_{u,v \in V(G)} d_G(u,v)$$

$$W(G_f) = \begin{matrix} d(u_1,u_2) + d(u_1,u_3) + d(u_1,u_4) + d(u_1,u_5) + d(u_2,u_3) + d(u_2,u_4) + \\ d(u_3,u_4) + d(u_2,u_5) + d(u_3,u_4) + d(u_3,u_5) + d(u_4,u_5) \end{matrix}$$

$$= (1+2+2+3) + (1+1+2) + (2+3) + 1$$

$$= (4 \times 1) + (4 \times 2) + (2 \times 3)$$

$$= (n-1) \times 1 + (n-2) \times 2 + (n-3) \times 3$$

$$\therefore \; W(G_f) = (n-1) \times 1 + (n-2) \times 2 + (n-3) \times 3$$

$$W(L(G_f)) =$$

$$d(u_1,u_2) + d(u_1,u_3) + d(u_1,u_4) + d(u_2,u_3) + d(u_2,u_4) + d(u_3,u_4)$$

$$= (1+2+2) + (1+1) + 1$$

$$= (4 \times 1) + (2 \times 2)$$

$$= n \times 1 + (n-2) \times 2$$

$$\therefore \; W(L(G_f)) = n \times 1 + (n-2) \times 2$$

Proposition : 4.4

The Weiner index of Cross graph is $W(L(G_c)) = (n+1) \times 1 + (n-4) \times 2 + (n-4) \times 3$

Proof:

Let G_c be a Fork graph with 5 vertices and 4 edges and $L(G_c)$ be the line graph of G_c with 5 vertices and 5 edges.

$$\therefore W(G) = \sum_{u,v \in V(G)} d_G(u,v)$$

$$W(G_c) = \begin{matrix} d(u_1,u_2) + d(u_1,u_3) + d(u_1,u_4) + d(u_1,u_5) + d(u_1,u_6) + d(u_2,u_3) + d(u_2,u_4) + \\ +d(u_2,u_5) + d(u_2,u_6) + d(u_3,u_4) + d(u_3,u_5) + d(u_3,u_6) + d(u_4,u_5) + d(u_4,u_6) + d(u_5,u_6) \end{matrix}$$

$$= (1+2+3+2+2) + (1+2+1+1) + (1+2+2) + (3+3) + 2$$

$$= (5 \times 1) + (7 \times 2) + (3 \times 3)$$

$$= (n-1)\times 1 + (n+1)\times 2 + (n-3)\times 3$$

$$W(L(G_c)) = \begin{matrix} d(u_1,u_2)+d(u_1,u_3)+d(u_1,u_4)+d(u_1,u_5)+d(u_2,u_3)+d(u_2,u_4)+ \\ +d(u_2,u_5)+d(u_3,u_4)+d(u_3,u_5)+d(u_4,u_5) \end{matrix}$$

$$= (1+1+1+3)+(1+2+3)+(1+2)+1$$

$$= (6\times 1)+(2\times 2)+(2\times 3)$$

$$= (n+1)\times 1 + (n-4)\times 2 + (n-4)\times 3$$

References

[1] Alhevaz, A., Baghipuri, M. and Shang, Y. (2019) On Generalized Distance Gaussian Estrada Index of Graphs. Symmetry, 11, 1276.

[2] Alhevaz, A., Baghipuri, M. and Shang, Y. (2019) On Generalized Distance Gaussian Estrada Index of Graphs. Symmetry, 11, 1276.

[3] Y.E.A. Ban, S. Berg and N.H. Mustafa. A Conjecture on Wiener Index in Combinatorial chemistry Algorithm 40(2): 99-118, 2004.

[4] D. Bonchev and D.J. Klein On the Wiener number of thron trees, stars, rings and rods, croatica chemical Acta 75:613-620, 2002.

[5] A.Dobriyan and I Gutman on a graph invariant related to sum of all distances in a graphpubl. Inst. Math. (Beograd) 56: 18-22 1994.

[6] H. Wiener Structural determination of the paraffin boiling points J. Am.Chem. Soc., 69 (1947), pp. 17-20

[7] Gutman, I. and Pavlovi´c, L., “More on distance of line graphs”, *Graph Theory Notes New York* **33** (1997), 14 - 18.

[8] Sreeji.S and S.S.Sandhya , Power 3 Mean Labeling of Line Graphs, Journal of Emerging Technologies and Innovative Research, Vol. VII,Issue :IV April -2021 ISSN NO:2349-5162.

[9] Walikar, H.B., Shigehalli, V.S. and Ramane, H.S. (2004) Bounds on the Wiener Indexof a Graph. MATCH Communications in Mathematical and in Computer Chemistry, 50, 117-132.

[10] Y.-N. Yeh, I. Gutman On the sum of all distances in composite graphs DiscreteMath., 135 (1994), pp. 359-365

[11] S. Sreeji, S.S. Sandhya, “Some New families of Power 3 Mean Graphs”, Journal of Information and Computational Science” ISSN N0: 1548-7741, Vol. 9, Issue 12, December – 2019

[12] S.Sreeji, S.S.Sandhya *Some More Results on Power 3 Mean Graphs*, International Journal of Scientific and Engineering Re- search ISSN NO: 2229-5518,Vol. 11, Issue 2, February-2020

[14] S.Sreeji, S.S.Sandhya *Power 3 Mean labeling in the context of Duplication of Graph Elements*, Journal of the American Mathematical Society, Advances in Mathematics: Scientific Journal 9 (2020), no.6, 4241-4249, ISSN 1857-8365.

[15] Zaslasvsky . T, A Mathematical Bibiliography of Signed and Gain Graphs and Allied Areas, Volume DS8 of Dynamic Surveys in Combinatorics, 1999.

[16] Zhao, Y & Wu, B 2015, *Star decomposition of graphs*, Discrete Mathematics Algorithms and Applications, vol. 7, no.2, pp.1-9.