Integral Solution of the Sextic Equation with Four Unknowns $x^3 + y^3 = 7zw^5$

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Abstract:

In this paper, we are going to solve the Sextic Diophantine equation with four unknowns. These types of equations are challenging for finding integer solutions. We will use various techniques such as the substitution method and ratio method to solve the Diophantine equations. Here we use the substitution method to find the integer solution of the given equation. Furthermore, our investigation has led us to uncover intriguing relationships between these solutions, which manifest in three distinct patterns.

Keywords:

Diophantine equations, Sextic equations, Integral sollutions.
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Introduction:

Today Diophantine analysis is the area of the study where integer solutions are sought for equations. The mathematical study of Diophantine problems that Diophantus initiated is now called Diophantine analysis. Diophantine equations are polynomial equations with their integer coefficients to which only integer solutions are sought. It is very difficult to identify whether a given Diophantine equation is solvable. Individual equations present a kind of puzzle and have been considered throughout history, the formulation of general theories of Diophantine equations was an achievement of the twentieth century.

The Diophantine equations offers an unlimited field for research due to their variety[1-4]. In particular one may refer [5-8] for Sextic equations with four unknowns. This communication concerns with yet another interesting equation $x^3 + y^3 = 7zw^5$ representing a non-homogeneous Sextic equation with four unknowns to determining its infinitely many non-zero integral points.

Method of analysis:

Problem:

The higher degree Diophantine equation is

$$x^3 + y^3 = 7zw^5 \tag{1}$$

Using the linear transformation,

$$x = u + v, \quad y = u - v, \quad z = 2u \tag{2}$$

Equation (1) reduces to

$$u^2 + 3v^2 = 7w^5 \tag{3}$$

Pattern:1

Let $w = w(a, b) = a^2 + 3b^2$ (4)

Write $7 = \left(2 + i\sqrt{3}\right)\left(2 - i\sqrt{3}\right)$ (5)

Using (4) and (5) in (3) and applying the method of factorization we get,

$$u + i\sqrt{3}v = \left(2 + i\sqrt{3}\right)\left(a + i\sqrt{3}b\right)^5 \tag{6}$$

Equating the real and imaginary parts, we have

$$\left.\begin{aligned} u(a,b) &= 2a^5 - 15a^4b - 60a^3b^2 + 90a^2b^3 + 90ab^4 - 27b^5 \\ v(a,b) &= a^5 + 10a^4b - 30a^3b^2 - 60a^2b^3 + 45ab^4 + 18b^5 \end{aligned}\right\} \tag{7}$$

Using equation (7) in (2) we get the integral solution of the given equation is

$$x(a,b) = 3a^5 - 5a^4b - 90a^3b^2 + 30a^2b^3 + 135ab^4 - 9b^5$$

$$x(a,b) = a^5 - 25a^4b - 30a^3b^2 + 150a^2b^3 + 45ab^4 - 45b^5$$

$$z(a,b) = 4a^5 - 30a^4b - 120a^3b^2 + 180a^2b^3 + 180ab^4 - 54b^5$$

$$w(a,b) = a^2 + 3b^2$$

Pattern:2

Write the number 7 as

$7 = \frac{1}{4}(1 + i3\sqrt{3})(1 - i3\sqrt{3})$ (8)

Following the procedure similar to pattern.1 we get,

$$u(a,b) = \frac{1}{2}[a^5 - 45a^4b - 30a^3b^2 + 270a^2b^3 + 45ab^4 - 81b^5]$$

$$v(a,b) = \frac{1}{2}[3a^5 + 5a^4b - 90a^3b^2 - 30a^2b^3 + 135ab^4 + 9b^5]$$

As our interest is to find non-zero distinct integral solution. Therefore replace a by 2a and b by 2b we get,

$$\left.\begin{aligned} u(a,b) &= 2^4[a^5 - 45a^4b - 30a^3b^2 + 270a^2b^3 + 45ab^4 - 81b^5] \\ v(a,b) &= 2^4[3a^5 + 5a^4b - 90a^3b^2 - 30a^2b^3 + 135ab^4 + 9b^5] \end{aligned}\right\} \quad (9)$$

Using (9) in (2) we get the non-zero distinct integral solution to equation (1) is,

$$x(a,b) = 2^4(4a^5 - 40a^4b - 120a^3b^2 + 240a^2b^3 + 180ab^4 - 72b^5)$$

$$y(a,b) = 2^4(-2a^5 - 50a^4b + 60a^3b^2 + 300a^2b^3 - 90ab^4 - 90b^5)$$

$$z(a,b) = 2^5(a^5 - 45a^4b - 30a^3b^2 + 270a^2b^3 + 45ab^4 - 81b^5)$$

$$w(a,b) = 2^2(a^2 + 3b^2)$$

Pattern:3

Write the number 7 as

$7 = \frac{(5*2^n + i2^n\sqrt{3})(5*2^n - i2^n\sqrt{3})}{2^{2n+2}}$ (10)

Following the procedure similar to pattern.1 we get,

$$u(a,b) = \frac{2^n}{2^{n+1}}(5a^5 - 150a^3b^2 + 225ab^4 - 15a^4b + 90a^2b^3 - 27b^5)$$

$$v(a,b) = \frac{2^n}{2^{n+1}}(a^5 - 30a^3b^2 + 45ab^4 + 25a^4b - 150a^2b^3 + 45b^5)$$

Using the values of u(a,b) and v(a,b) in equation (2) we get the non-zero distinct integral solution to (1) is given by

$$x(a,b) = \frac{2^n}{2^{n+1}}(6a^5 - 180a^3b^2 + 170ab^4 + 10a^4b - 60a^2b^3 + 18b^5)$$

$$y(a,b) = \frac{2^n}{2^{n+1}}(4a^5 - 120a^3b^2 - 180ab^4 - 40a^4b + 240a^2b^3 - 72b^5)$$

$$z(a,b) = (5a^5 - 150a^3b^2 + 225ab^4 - 15a^4b + 90a^2b^3 - 27b^5)$$

$w(a,b) = a^2 + 3b^2$ Where n = 0,1,2……….

Conclusion:

Solving the Diophantine equation and finding the non-zero distinct integer solutions are used in the field of Chemistry: the balancing of chemical equations and determining the molecular formula of a compound. It is also used in various fields like cryptography, Number patterns. The concepts of Diophantine equations are encouraging the young researchers to discover new ideas in various fields like some mentioned above. To conclude one may search for different types of Diophantine equations and their solutions.

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