GENERALIZATION OF CONTRA HARMONIC MEAN AND THEIR PROPERTIES

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Abstract: The aim of this paper is to understand the existing results on generalization of well known means in the field of classical mathematical means and to contribute some new results on monotonicities.

Keywords: Contra harmonic mean, monotonicity, generalization, Heron mean.

1. INTRODUCTION and LITERATURE SURVEY

In recent years, pure and applied mathematics have seen several notable trends, reflecting advancements in technology, changes in research focus, and the evolving landscape of scientific inquiry. In pure mathematics, there has been a growing emphasis on interdisciplinary collaboration, with mathematicians working closely with researchers in other fields such as physics, computer science, and biology.

Pappas of Alexandria in his books during the fourth century A.D. presented the well known means which are the main contributions of the ancient Greeks. In Pythagorean School on the basis of proportion ten Greek means are defined of which well known means $A(a,b)=\frac{a+b}{2}$, $G(a,b)=\sqrt{ab}$, $H(a,b)=\frac{2ab}{a+b}$, and $C(a,b)=\frac{a^2+b^2}{a+b}$, are respectively called Arithmetic mean, Geometric mean, Harmonic mean and Contra harmonic mean [1,4]. The arithmetic mean of two positive numbers a and b, the number 'm' such that $m-a:b-m::1:1$ on simplifying gives $m=\frac{a+b}{2}$ is the form of arithmetic mean which is one of the ten Neo-Pythagorean means (Greek means) [1,2,4] .

The generalization of Arithmetic mean, Geometric mean, and Harmonic mean is given by [3,13,14,15,16,19]:

$$M_r(a,b)=\begin{cases}\left(\dfrac{a^r+b^r}{2}\right)^{1/r} & r\neq 0\\ \sqrt{ab} & r=0\end{cases}$$

which is also known as power mean.

The linear combination of Arithmetic mean and Geometric mean is also called Heron mean denoted and defined [4] as $H_e(a,b)=\frac{a+\sqrt{ab}+b}{3}$, which occurs in an Egyptian manuscript in the year 1850 B.C. the Moscow Papyrus presented the volume in terms of Heron mean in the form of $V=hH_e(a,b)$.Where $H_e(a,b)$ is the Heron mean of '*a*' and '*b*', *h* is the height of the frustum of pyramid, '*a*' lower base area and '*b*' upper base area.

G. Toader and S. Toader [4], gave the brief collection of ten Greek means, comparison of means, weighted means, invariant and complimentary means and partial derivatives of means and its related results; further double sequences are defined in terms of means and some applications of double sequences to estimate the value of 'π' and extraction square roots were established by Heron. Lagrange's Gaussian and Archimedean double sequences were defined and some related results are mentioned. Bullen introduced particular cases of extended means in his book [1]. Zhen-Hang Yang in his papers [6-9], defined homogeneous function with two parameters $H_f(a, b; p, q)$ and proved its monotonicity with respect to *p* and *q* as a general case by using Lagrange's mean value theorem. Also he considered four particular cases; namely,

1. $f(x,y)=A(x,y)=\dfrac{x+y}{2}$,
2. $f(x,y)=L(x,y)=\begin{cases}\dfrac{x-y}{\ln x-\ln y} & x\neq y\\ x & x=y\end{cases}$,
3. $f(x,y)=I(x,y)=\begin{cases}e^{\left(\frac{x\ln x-y\ln y}{x-y}-1\right)} & x\neq y\\ x & x=y\end{cases}$
4. $f(x,y)=D(x,y)=|x-y|$

where $(x, y) = (a^p, b^p)$ or (a^q, b^q) and established valuable inequality chains to generalize, strengthen and unify the old inequalities. Several estimates for lower and upper bounds of two parameter logarithmic mean (extended mean) are presented. Further, author studied logarithmical convexity of homogeneous function, refinements and extension of logarithmical convexity[17,18], some identities and applications of $H_f(a, b; p, q)$. Also author introduced the power exponential mean. Several authors obtained an excellent work on monotonicities by L'Hospital rule and some remarkable results on Mathematical means and its inequalities [5,10,11,12].

In [13] the generalized α - centroidal Mean and its dual form in 2 variables are introduced. Also, studied some properties and proved their monotonicity. Further, shown that various means are particular cases of generalized α - centroidal Mean. In [17] obtained some results and generalization of the contra harmonic mean in several variables. Further, introduced Stolarsky's extended family type mean values and studied the properties and monotonic results. In [8] the log convexity and a simple proof for monotonicity of power mean and generalized contra-harmonic mean are presented in the literature.

2. DEFINITIONS

In this section, few essential definitions which are required to develop this paper are recalled and are well known in literature.

Definition 2.1: Convolution of sequences is an operation of multiplication that in most applications of sequences. It is taken with the same priority as multiplication in algebraic calculations: In particular, it has higher priority than addition of sequences[12].

It is usually denoted by a star.

$$(f * g)(n) = \sum_{m=-\infty}^{m=\infty} f(m)\, g(n-m) = \sum_{m=-\infty}^{m=\infty} f(n-m)\, g(m)$$

Definition 2.2: From above definition it generates for the sequences[12]; $f(m) = a^m$ and $g(k-m) = b^{k-m}$, then

$$\sum_{m=0}^{K} a^m b^{K-m} = \frac{a^{K+1} - b^{K+1}}{a - b}$$

Definition 2.3: For $a, b > 0$, K be any positive integer, then the following generalized forms of Heron mean and its dual are constructed. Further, studied their various properties.

$$\mathrm{H_n}(a, b, K, \alpha, \beta) = \frac{1}{K+1}\left[\sum_{i=0}^{K}\left(\frac{(K-i)\,a^{\alpha} + i\,b^{\alpha}}{K}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \quad \text{for } \alpha \neq \beta$$

and its dual as,

$$\mathrm{H_n^{(d)}}(a, b, K, \alpha, \beta) = \frac{1}{K}\left[\sum_{i=1}^{K}\left(\frac{(K+1-i)\,a^{\alpha} + i\,b^{\alpha}}{K+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \quad \text{for } \alpha \neq \beta$$

This work motivates to construct the generalized form of centroidal mean in this paper.

3. GENERALIZATION OF MEANS

This section provides, the generalized form of contra harmonic mean & their dual.

Definition 3.1: For $a, b > 0$, the centroidal mean is given by;

$$CH(a, b) = \frac{a^2 + b^2}{a + b}$$

As above, for $a, b > 0$, non-negative integer K, define the generalized contra harmonic mean $CH(a, b, K)$ as;

$$CH(a, b, K) = \frac{\sum_{i=0}^{K} a^{\frac{2(K-i)}{K}} b^{\frac{2i}{K}}}{\sum_{i=0}^{K} a^{\frac{(K-i)}{K}} b^{\frac{i}{K}}} \quad for\ K \geq 2 \qquad \text{---(1)}$$

and its dual as,

$$CH^{(d)}(a, b, K) = \frac{\sum_{i=1}^{K} a^{\frac{2(K+1-i)}{K+1}} b^{\frac{2i}{K+1}}}{\sum_{i=1}^{K} a^{\frac{(K+1-i)}{K+1}} b^{\frac{i}{K+1}}} \quad for\ K \geq 2 \qquad \text{---(2)}$$

OR

$$CH^{(d)}(a, b, K) = \frac{\sum_{i=0}^{K} a^{\frac{2(K+1-i)}{K}} b^{\frac{2i}{K+1}} - a^2}{\sum_{i=0}^{K} a^{\frac{(K-i)}{K}} b^{\frac{i}{K}} - a} \quad for\ K \geq 2 \qquad \text{---(2)}$$

Definition 3.2: For $a, b > 0$, K be any positive integer and α, β are 2 real numbers, then generalization of contra harmonic mean is given by;

$$\mathrm{CH}(a,b,K\colon \alpha,\beta) = \frac{\left[\sum_{i=0}^{K}\left(\frac{2(K-i)\,a^{\alpha} + 2i\,b^{\alpha}}{K}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}}{\left[\sum_{i=0}^{K}\left(\frac{(K-i)\,a^{\alpha} + i\,b^{\alpha}}{K}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}}\ ,\alpha \neq \beta$$

and its dual

$$\mathrm{CH}^{(\mathrm{d})}(a,b,K\colon \alpha,\beta) = \frac{\left[\sum_{i=1}^{K}\left(\frac{2(K+1-i)\,a^{\alpha} + 2i\,b^{\alpha}}{K}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}}{\left[\sum_{i=1}^{K}\left(\frac{(K+1-i)\,a^{\alpha} + i\,b^{\alpha}}{K}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}}\ ,\alpha \neq \beta$$

4. RESULTS

Through this section stated properties and proved the monotonicities.

Properties 4.1: Obviously $CH(a,b,K\colon \alpha,\beta)$ and $CH^{(d)}\,(a,b,K\colon \alpha,\beta)$ are symmetric and homogeneous in nature.

Moonotonicity 4.2: For $a,b > 0$, K is any positive integer, then the generalized contra harmonic mean $CH(a,b,K)$ is monotonically decreasing in nature.

Proof: Consider, $CH(a,b,K) = \dfrac{\sum_{i=0}^{K} a^{\frac{2(K-i)}{K}} b^{\frac{2i}{K}}}{\sum_{i=0}^{K} a^{\frac{(K-i)}{K}} b^{\frac{i}{K}}} = \dfrac{\sum_{i=0}^{K}\left(a^{\frac{2}{K}}\right)^{(K-i)}\left(b^{\frac{2}{K}}\right)^{i}}{\sum_{i=0}^{K}\left(a^{\frac{1}{K}}\right)^{(K-i)}\left(b^{\frac{1}{K}}\right)^{i}}$

$$= \frac{\dfrac{\left(a^{\frac{2}{K}}\right)^{(K+1)} - \left(b^{\frac{2}{K}}\right)^{(K+1)}}{\left(a^{\frac{2}{K}}\right) - \left(b^{\frac{2}{K}}\right)}}{\dfrac{\left(a^{\frac{1}{K}}\right)^{(K+1)} - \left(b^{\frac{1}{K}}\right)^{(K+1)}}{\left(a^{\frac{1}{K}}\right) - \left(b^{\frac{1}{K}}\right)}} = \frac{a^{\frac{K+1}{K}} + b^{\frac{K+1}{K}}}{a^{\frac{1}{K}} + b^{\frac{1}{K}}}$$

In the above expression replace *K* by *K+1* gives;

$$CH(a,b,K+1) = \frac{a^{\frac{K+2}{K+1}} + b^{\frac{K+2}{K+1}}}{a^{\frac{1}{K+1}} + b^{\frac{1}{K+1}}}$$

Consider, $CH(a,b,K+1) - CH(a,b,K) = \frac{a^{\frac{K+2}{K+1}}+b^{\frac{K+2}{K+1}}}{a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}} - \frac{a^{\frac{K+1}{K}}+b^{\frac{K+1}{K}}}{a^{\frac{1}{K}}+b^{\frac{1}{K}}}$

Further on simplification leads to: $= \frac{a^{\frac{K+2}{K+1}}+b^{\frac{K+2}{K+1}}}{a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}} - \frac{a^{\frac{K+1}{K}}+b^{\frac{K+1}{K}}}{a^{\frac{1}{K}}+b^{\frac{1}{K}}} =$

$$\frac{(a-b)\left(a^{\frac{1}{K+1}}b^{\frac{1}{K}}-a^{\frac{1}{K}}b^{\frac{1}{K+1}}\right)}{\left(a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}\right)\left(a^{\frac{1}{K}}+b^{\frac{1}{K}}\right)} \quad \text{-----(3)}$$

Case(i) For any positive integer K and $b > a$,

It was well known that: $\left(\frac{a}{b}\right)^{K} \geq \left(\frac{a}{b}\right)^{K+1}$

Equivalently, $\left(\frac{a}{b}\right)^{\frac{1}{K}} \geq \left(\frac{a}{b}\right)^{\frac{1}{K+1}}$

Or $a^{\frac{1}{K}}b^{\frac{1}{K+1}} \geq b^{\frac{1}{K}}a^{\frac{1}{K+1}}$ or $a^{\frac{1}{K}}b^{\frac{1}{K+1}} - b^{\frac{1}{K}}a^{\frac{1}{K+1}} \geq 0.$

Therefore, $(a-b)\left(a^{\frac{1}{K+1}}b^{\frac{1}{K}}-a^{\frac{1}{K}}b^{\frac{1}{K+1}}\right) = (-ve) * (+ve) < 0.$

Case(ii) For any positive integer K and $a > b$,

It was well known that: $\left(\frac{a}{b}\right)^{K} \leq \left(\frac{a}{b}\right)^{K+1}$

Equivalently, $\left(\frac{a}{b}\right)^{\frac{1}{K}} \leq \left(\frac{a}{b}\right)^{\frac{1}{K+1}}$

Or $a^{\frac{1}{K}}b^{\frac{1}{K+1}} \leq b^{\frac{1}{K}}a^{\frac{1}{K+1}}$ or $a^{\frac{1}{K}}b^{\frac{1}{K+1}} - b^{\frac{1}{K}}a^{\frac{1}{K+1}} \leq 0.$

Therefore, $(a-b)\left(a^{\frac{1}{K+1}}b^{\frac{1}{K}}-a^{\frac{1}{K}}b^{\frac{1}{K+1}}\right) = (+ve) * (-ve) < 0.$

By considering case(i) and case(ii) proves that eqn(3) is negative.

Hence the generalized contra harmonic mean $CH(a,b,K)$ is monotonically decreasing in nature for all values of a, b & K.

Moonotonicity 4.3: For $a, b > 0, K$ be any positive integer, then the generalized contra harmonic mean $CH^{(d)}(a,b,K)$ is monotonically increasing in nature.

Proof: Consider $CH^{(d)}(a,b,K) = \frac{\sum_{i=0}^{K}\left(a^{\frac{2(K+1-i)}{K+1}}b^{\frac{2i}{K+1}}\right)-a^2}{\sum_{i=0}^{K}\left(a^{\frac{(K+1-i)}{K+1}}b^{\frac{i}{K+1}}\right)-a} =$

$$\frac{\sum_{i=0}^{K}\left[\left(a^{\frac{2}{K+1}}\right)^{(K+1-i)}\left(b^{\frac{2}{K+1}}\right)^{i}\right]-a^2}{\sum_{i=0}^{K}\left[\left(a^{\frac{1}{K+1}}\right)^{(K+1-i)}\left(b^{\frac{1}{K+1}}\right)^{i}\right]-a}$$

$$= \frac{\left(\frac{\left(a^{\frac{2}{K+1}}\right)^{k+2} - \left(b^{\frac{2}{K+1}}\right)^{k+2}}{\left(a^{\frac{2}{K+1}}\right) - \left(b^{\frac{2}{K+1}}\right)}\right) - a^2}{\left(\frac{\left(a^{\frac{1}{K+1}}\right)^{k+2} - \left(b^{\frac{1}{K+1}}\right)^{k+2}}{\left(a^{\frac{1}{K+1}}\right) - \left(b^{\frac{1}{K+1}}\right)}\right) - a}$$

On simplification leads to:

$$= \left(\frac{a^2+b^2}{a+b}\right)\left(\frac{b^{\frac{1}{K+2}}}{a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}}\right)$$

In the above expression replace K by $K+1$ gives;

$$CH(a,b,K+1) = \left(\frac{a^2+b^2}{a+b}\right)\left(\frac{b^{\frac{1}{K+2}}}{a^{\frac{1}{K+2}}+b^{\frac{1}{K+2}}}\right)$$

Consider, $CH(a,b,K+1) - CH(a,b,K) = \left(\frac{a^2+b^2}{a+b}\right)\left[\left(\frac{b^{\frac{1}{K+2}}}{a^{\frac{1}{K+2}}+b^{\frac{1}{K+2}}}\right) - \left(\frac{b^{\frac{1}{K+2}}}{a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}}\right)\right]$

Further simplification leads to:

$$= \left(\frac{a^2+b^2}{a+b}\right)\left[\frac{a^{\frac{1}{K+1}}b^{\frac{1}{K+2}} - b^{\frac{1}{K+1}}a^{\frac{1}{K+2}}}{\left(a^{\frac{1}{K+2}}+b^{\frac{1}{K+2}}\right)\left(a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}\right)}\right] \qquad \text{-----(3)}$$

Case(i) For any positive integer K and $b > a$,

It was well known that: $\left(\frac{a}{b}\right)^{K+1} \geq \left(\frac{a}{b}\right)^{K+2}$

Equivalently, $\left(\frac{a}{b}\right)^{\frac{1}{K+2}} \leq \left(\frac{a}{b}\right)^{\frac{1}{K+1}}$

Or $\quad b^{\frac{1}{K+1}}a^{\frac{1}{K+2}} \leq a^{\frac{1}{K+1}}b^{\frac{1}{K+2}}$ or $b^{\frac{1}{K+1}}a^{\frac{1}{K+2}} - a^{\frac{1}{K+1}}b^{\frac{1}{K+2}} \leq 0.$

Therefore, $\left(\frac{a^2+b^2}{a+b}\right)\left[\frac{a^{\frac{1}{K+1}}b^{\frac{1}{K+2}} - b^{\frac{1}{K+1}}a^{\frac{1}{K+2}}}{\left(a^{\frac{1}{K+2}}+b^{\frac{1}{K+2}}\right)\left(a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}\right)}\right] = (+ve) * (-ve) < 0.$

Case(ii) For any positive integer K and $a > b$,

It was well known that: $\left(\frac{a}{b}\right)^{K+1} \geq \left(\frac{a}{b}\right)^{K+2}$

Equivalently, $\left(\frac{a}{b}\right)^{\frac{1}{K+2}} \leq \left(\frac{a}{b}\right)^{\frac{1}{K+1}}$

Or $\quad b^{\frac{1}{K+1}}a^{\frac{1}{K+2}} \leq a^{\frac{1}{K+1}}b^{\frac{1}{K+2}}$ or $b^{\frac{1}{K+1}}a^{\frac{1}{K+2}} - a^{\frac{1}{K+1}}b^{\frac{1}{K+2}} \leq 0.$

Therefore, $\left(\frac{a^2+b^2}{a+b}\right)\left[\frac{a^{\frac{1}{K+1}}b^{\frac{1}{K+2}} - b^{\frac{1}{K+1}}a^{\frac{1}{K+2}}}{\left(a^{\frac{1}{K+2}}+b^{\frac{1}{K+2}}\right)\left(a^{\frac{1}{K+1}}+b^{\frac{1}{K+1}}\right)}\right] = (+ve) * (-ve) < 0.$

By considering case(i) and case(ii) proves that eqn(3) is negative.

Hence the generalized contra harmonic mean $CH(a, b, K)$ is monotonically decreasing in nature for all values of a, b & K.

References:

1. P. S. Bullen, Handbook of Means and Their Inequalities, Mathematics and its Applications (Dordrecht) 560, Kluwer Academic Publishers, Dordrecht, 2003.
2. G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, 2nd edition, CambridgeUniversity Press, Cambridge, 1959.University Press, Cambridge, 1959**.**
3. V. Lokesha and Zhi-Hua Zhang, The weighted heron dual mean in n variables, Adv. Stud. Contemp. Math., 13(2), (2006), pp. 165-170.
4. G. Toader and S. Toader, Greek means and Arithmetic and Geometric means, RGMIA monograph, Australia, 2005. http://rgmia.vu.edu.au/html.
5. R. Webster, Convexity, Oxford University Press, Oxford, New York, Tokyo, 1994.
6. Zhen-Gang Xiao and Zhi-Hua Zhang, The Inequalities G < L < I < A in *n* variables, J. Ineq. Pure and Appl. Math., 4(2), (2003), article 39. http://jipam.vu.edu.au/v4n2/110 02.pdf
7. Zhen-Hang Yang, On the Homogeneous function with two parameters and its monotonicity,RGMIA research report Collection, 8(2), article 10, (2005).
8. Zhen-Hang Yang, On the log-convexity for two parameters Homogeneous function, RGMIA research report Collection, 8(2), article 21, (2005), Australia. http://rgmia.vu.edu.au/v8n2.html.
9. Zhen-Hang Yang, On refinements and extensions of log-convexity for two parameters Homogeneous function, RGMIA research report Collection, 8(3), article 12, (2005), Australia. http://rgmia.vu.edu.au/v8n3.html.
10. Zhen-Hang Yang, Some identities for means and applications, RGMIA research report Collection, 8(3), article. 17, (2005), Australia. http://rgmia.vu.edu.au/v8n3.html.
11. E. Smith, Characterizing Computer Performance with a single number, Commun, ACM, 31(10), (1988), pp. 1202-1206.
12. Lokesha, V., & Nagaraja, K. M. (2007). Relation between series and important means. *Advances in theoretical and applied mathematics*, *2*(1), 31-37.
13. Nagaraja, K. M., SIVA, K. R. P., & Sahu, S. K. (2013). Generalization of α-Centroidal Mean and its Dual.
14. Nagaraja, K. M., Reddy, P. S. K., & SAHU, S. K. (2012). Stolarsky's extended family type mean values. *Bull. of Pure & Appl. Math*, *6*(2), 255-264.
15. Nagaraja, K. M., & Reddy, P. S. K. (2012). A Note on power mean and generalized contra-harmonic mean. *Scientia Magna*, *8*(3), 60-62.
16. Lokesha, V., Nagaraja, K. M., Kumar, N., & Wu, Y. D. (2011). Schur convexity of Gnan mean for two variables. *NNTDM*, *17*(4), 37-41.
17. Nagaraja, K. M., & Sahu, S. K. (2013). Schur harmonic convexity of Stolarsky extended mean values. *Scientia Magna*, *9*(2), 18.
18. Nagaraja, K. M., Reddy, P. S. K., & Sridevi, K. (2013). Schur harmonic convexity of Gnan

mean for two variables. *Journal of the International Mathematical Virtual Institute*, *3*, 61-80.

19. Nagaraja, K. M., & Sahu, S. K. (2013). Schur Geometric convexity of Gnan mean for two variables. *Journal of the International Mathematical Virtual Institute*, *3*, 39-59.