Approximate Fixed Point for Closed Graph Operator Satisfying Contractive Conditions

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Abstract

The present article emphasizes a unique fixed point which is proved by contractive type conditions that are drawn by a closed graph operator of a complete metric space such as Kannan and Chatterjea. To satisfy the contractive condition, we have considered a self-map of a non-decreasing sequence of subsets of a complete metric space, hence a contractive condition is satisfied.

Keywords: approximate fixed point, closed graph, Kannan and Chatterjea mappings.

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1 Introduction

A self map of a metric space (X, d) into itself is said to have a closed graph, if whenever $x_n \to x_0$ and $Tx_n \to y_0$, for some sequences $\{x_n\}$ in X and some $x_0, y_0 \in X$. Some fixed point theorems in "Contraction mapping with variations in domain", was developed by C. Ganesa Moorthy and P. Xavier Raj [1]. Zamfirescu[3], who is the first personality, had combined the proved states of Banach, Kannan and Chatterjea's setup to the concept of some fixed point theorems. The paper has drawn by the extension of Kannan and Chatterjea strategies of some contraction mapping of a closed graph which is to establish some fixed point theorems.

2 Preliminaries

Definition 2.1. *[3] For a mapping $T : (X, d) \to (X, d)$, there exist real numbers α, β, γ satisfying $0 < \alpha < 1$, $0 < \beta, \gamma < \frac{1}{2}$ such that for each $p, q \in X$, at least one of the* *following is true.*

(i) $d(Tp, Tq) \leq \alpha d(p, q)$

(ii) $d(Tp, Tq) \leq \beta [d(p, Tp) + d(q, Tq)]$

(iii) $d(Tp, Tq) \leq \gamma [d(p, Tq) + d(q, Tp)]$

Theorem 2.2 ([1]). *Let (X, d) be a complete metric space and $T : X \to X$ have a closed graph. Let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ be subsets of X such that $T(X_i) \subseteq X_{i+1}$, for all i, $X = \bigcup_{j=1}^{\infty} X_j$ and $d(Tx, Ty) \leq k_i d(x, y)$, for all $x, y \in X_i$, where $k_i \in (0, 1)$ are constants such that $\sum_{n=1}^{\infty} k_1, k_2, \ldots, k_n < \infty$. Then T has a unique fixed point in X.*

3 Main Results

Theorem 3.1. *Let $\{X_i\}$ be subsets of a complete metric space (X, d) such that $X_i \subseteq X_{i+1}$ and $X = \bigcup_{i=1}^{\infty} X_i$. Suppose $T : X \to X$ be a closed graph operator satisfying $T(X_i) \subseteq X_{i+1}$, for every i, and one of the following conditions is satisfied:*

(i) $d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)]$,

(ii) $d(Tx, Ty) \leq \gamma [d(x, Ty) + d(y, Tx)]$, *for every* $x, y \in X_i$,

where $0 < \beta, \gamma < \frac{1}{2}$, *such that* $\sum_{n=1}^{\infty} \frac{\alpha_1}{1-\alpha_1} \frac{\alpha_2}{1-\alpha_2} \cdots \frac{\alpha_n}{1-\alpha_n} < \infty$. *Then T has a unique fixed point in X.*

Proof : (i) Fix $x_1 \in X_1$. Define $x_{n+1} = Tx_n = T(Tx_{n-1}) = T^n x_1$ for all $n = 1, 2, \ldots$. Then

$$d(x_2, x_1) = d(Tx_1, x_1)$$

Now,

$$\begin{aligned} d(x_3, x_2) &= d(Tx_2, Tx_1) \\ &\leq \alpha_1 [d(x_2, Tx_2) + d(x_1, Tx_1)] \\ &\leq \alpha_1 [d(x_2, x_2) + d(x_1, x_2)] \\ (1-\alpha_1) d(x_3, x_2) &= \alpha_1 d(x_1, x_2) \\ d(T^2 x_1, Tx_1) &\leq \frac{\alpha_1}{1-\alpha_1} d(x_1, Tx_1) \end{aligned}$$

Similarly,

$$d(T^3x_1, T^2x_1) \leq \frac{\alpha_2}{1-\alpha_2} \cdot \frac{\alpha_1}{1-\alpha_1} d(x_1, Tx_1)$$

In general, we get,

$$d(T^{n+1}x_1, T^nx_1) \leq \frac{\alpha_n}{1-\alpha_n} \cdots \frac{\alpha_2}{1-\alpha_2} \frac{\alpha_1}{1-\alpha_1} d(x_1, Tx_1)$$

For $1 \leq m < n$, we have,

$$\begin{aligned} d(T^mx_1, T^nx_1) &\leq d(T^mx_1, T^{m+1}x_1) + d(T^{m+1}x_1, T^{m+2}x_1) + \cdots + d(T^{n-1}x_1, T^nx_1) \\ &\leq \frac{\alpha_m}{1-\alpha_m} \cdots \frac{\alpha_2}{1-\alpha_2} \frac{\alpha_1}{1-\alpha_1} d(x_1, Tx_1) \\ &\quad + \frac{\alpha_{m+1}}{1-\alpha_{m+1}} \cdots \frac{\alpha_2}{1-\alpha_2} \frac{\alpha_1}{1-\alpha_1} d(x_1, Tx_1) + \ldots \\ &\quad + \frac{\alpha_{n-1}}{1-\alpha_{n-1}} \cdots \frac{\alpha_2}{1-\alpha_2} \frac{\alpha_1}{1-\alpha_1} d(x_1, Tx_1) \\ &= \sum_{i=m}^{n-1} \frac{\alpha_i}{1-\alpha_i} \cdots \frac{\alpha_2}{1-\alpha_2} \frac{\alpha_1}{1-\alpha_1} d(x_1, Tx_1) \end{aligned}$$

Thus, $\{T^nx_1\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Assume that $\{T^nx_1\}_{n=1}^{\infty}$ converges to s^* in X.

Then, $\{T^{n+1}x_1\}_{n=1}^{\infty}$ is also a Cauchy sequence and converges to s^* in X.

Since T has a closed graph in X, we have $Ts^* = s^*$, then s^* is a fixed point of T.

If r^* is another fixed point of T, then $s^*, r^* \in X$, so that

$$\begin{aligned} 0 &\leq d(s^*, r^*) = d(Ts^*, Tr^*) \\ &\leq \alpha_i[d(s^*, Tx^*) + d(r^*, Tr^*)] = 0 \\ 0 &\leq d(s^*, r^*) \leq 0 \end{aligned}$$

So, $s^* = r^*$, which proves the uniqueness of the fixed point of T.

(ii) Fix $x_1 \in X_1$. Define $x_{n+1} = Tx_n = T^nx_1$ for all $n = 1, 2, 3, \ldots$. Then, $d(x_2, x_1) = d(Tx_1, x_1)$ and

$$\begin{aligned} d(x_3, x_2) &= d(Tx_2, Tx_1) \\ &\leq \alpha_1 \left[d(x_2, Tx_1) + d(x_1, Tx_2)\right] \\ &\leq \alpha_1 \left[d(x_2, x_2) + d(x_1, x_3)\right] \\ &\leq \alpha_1 \left[d(x_1, x_2) + d(x_2, x_3)\right] \\ (1-\alpha_1)d(x_2, x_3) &\leq \alpha_1 d(x_1, x_2) \\ d(T^2x_1, Tx_1) &\leq \frac{\alpha_1}{1-\alpha_1} d(x, Tx) \end{aligned}$$

In a similar manner, we get

$$d(T^3x_1, T^2x_1) \leq \frac{\alpha_2}{1-\alpha_2}\frac{\alpha_1}{1-\alpha_1}d(x_1, Tx_1)$$

Continuing in this way we get,

$$d(T^{n+1}x_1, T^nx_1) \leq \frac{\alpha_n}{1-\alpha_n}\frac{\alpha_{n-1}}{1-\alpha_{n-1}}\cdots\frac{\alpha_1}{1-\alpha_1}d(x_1, Tx_1)$$

For $1 \leq m < n$, we have

$$\begin{aligned} d(T^mx_1, T^nx_1) &\leq d(T^mx_1, T^{m+1}x_1) + d(T^{m+1}x_1, T^{m+2}x_1) + \cdots + d(T^{n-1}x_1, T^nx_1) \\ &\leq \frac{\alpha_m}{1-\alpha_m}\frac{\alpha_{m-1}}{1-\alpha_{m-1}}\cdots\frac{\alpha_1}{1-\alpha_1}d(x_1, Tx_1) + \cdots \\ &\quad + \frac{\alpha_{n-1}}{1-\alpha_{n-1}}\frac{\alpha_{n-2}}{1-\alpha_{n-2}}\cdots\frac{\alpha_1}{1-\alpha_1}d(x_1, Tx_1) \\ d(T^mx_1, T^nx_1) &\leq \sum_{i=m}^{n-1}\frac{\alpha_1}{1-\alpha_1}\frac{\alpha_2}{1-\alpha_2}\cdots\frac{\alpha_i}{1-\alpha_i}d(x_1, Tx_1) \end{aligned}$$

$\{T^nx_1\}_{n=1}^{\infty}$ is a Cauchy sequence in X which converges to a point s^* in X. Then $\{T^{n+1}x_1\}_{n=1}^{\infty}$ is also a Cauchy sequence which converges to s^* in X.

Since T has a closed graph operator in X, we say that s^* is a fixed point of T.

Uniqueness part.

If r^* is a fixed point of T. Then s^* and r^* in X, so that

$$\begin{aligned} 0 \leq d(s^*, r^*) &= d(Ts^*, Tr^*) \\ &\leq \alpha_i[d(s^*, Tr^*) + d(r^*, Ts^*)] \\ &\leq \alpha_i[d(s^*, r^*) + d(r^*, s^*)] \\ &= 2\alpha_i d(s^*, r^*) \\ (1-2\alpha_i)d(s^*, r^*) &\leq 0 \end{aligned}$$

which implies that $d(s^*, r^*) = 0$ as $1-2\alpha_i < 0$. Hence, s^* is a unique fixed point of T. □

Example 3.2. Let $\alpha_n = \frac{1}{3}$, $n = 1, 2, 3, \ldots$, then

$$\sum_{n=1}^{\infty}\frac{\alpha_1}{1-\alpha_1}\frac{\alpha_2}{1-\alpha_2}\cdots\frac{\alpha_n}{1-\alpha_n} < \infty.$$

Let $X = \left\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\right\}$ and d be the usual metric on X.

Let us define:

$$X_n = \left\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots, \frac{1}{3^{n+1}}\right\},$$

for each $n = 1, 2, 3, \ldots$.

Define $T : X \to X$ by

$$T(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{3n+2}, & \text{if } x = \frac{1}{n}, n = 1, 2, 3, \ldots \end{cases}$$

For $p < q$, we have:

$$\begin{aligned}
\left| T\left(\frac{1}{p}\right) - T\left(\frac{1}{q}\right) \right| &= \left| \frac{1}{3^{p+2}} - \frac{1}{3^{q+2}} \right| \\
&= \left| \frac{1}{3^{p+2}} - \frac{1}{3^{q+1}} + \frac{1}{3^{q+1}} - \frac{1}{3^{q+2}} \right| \\
&\leq \frac{1}{3} \left\{ \left| \frac{1}{3^{p+1}} - \frac{1}{3^{q}} \right| + \left| \frac{1}{3^{q}} - \frac{1}{3^{q+1}} \right| \right\} \\
&= \frac{1}{3} \left\{ \left| \frac{1}{3^{q}} - \frac{1}{3^{p+1}} \right| + \left| \frac{1}{3^{q}} - \frac{1}{3^{q+1}} \right| \right\} \\
&< \frac{1}{3} \left\{ \left| \frac{1}{3^{p}} - \frac{1}{3^{p+1}} \right| + \left| \frac{1}{3^{q}} - \frac{1}{3^{q+1}} \right| \right\} \\
&< \frac{1}{3} \left\{ \left| \frac{1}{3^{p}} - \frac{1}{3^{p+2}} \right| + \left| \frac{1}{3^{q}} - \frac{1}{3^{q+2}} \right| \right\}
\end{aligned}$$

which is verified by the condition (i) of the above Theorem 3.1 also the fixed point is 0. In a similar fashion, condition (ii) is also proved.

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