(3 , 2)-FUZZY SUB ALGEBRAS AND ITS APPLICATIONS

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Abstract: In this paper, the concept of (3,2)- fuzzy set is investigated and compared with other types of uncertainty sets. We define Pythagorean fuzzy sub algebras in BCK-algebras and BCI-algebras and study their properties. A given (3,2)- fuzzy sub algebra is used to create a new (3,2)- fuzzy sub algebra. Also we obtain the intersection of two (3,2)- fuzzy sub algebras to be (3,2)- fuzzy sub algebra is proved and an example is given to show that the union of (3,2)- fuzzy sub algebras may not be (3,2)- fuzzy sub algebra. The characterization of cut set is also used in (3,2)- fuzzy sub algebra. The homomorphic image and pre image of (3,2)- fuzzy sub algebra is discussed. It turns out that Pythagorean fuzzy sub algebra is a subclass of (3,2)- fuzzy sub algebra.

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1.Introduction

An intuitionistic fuzzy set (IFS) is a set developed to handle problems related to imprecise and incomplete information [7]. This set was introduced by Atanassov, which is a generalization of the fuzzy set (FS) theory [30]. In FS, an element is marked by the presence of its membership (M)degree or value (i.e., the non-membership (N) degree is directly complemented to it). Mean while,in IFS, it is indicated by the presence of its M and N degrees, where the sum of the two can be less than one (i.e., any hesitancy or incomplete information is allowed). This makes IFS more flexible and covers more uncertain events in the decision-making process. Several studies have been conducted to expand the IFS, including in aggregation operators [26], and correlation coefficient [17], to mention a few. In addition, many authors have applied the IFS to decision-making problems [1, 13].IFS has experienced numerous developments, especially in terms of the relationship between M and N degrees. Initially, the IFS met the condition $M + N \leq 1$. However, to cater for the issue beyond this inequality (i.e., $M + N > 1$), Yager [28] then defined the Pythagorean fuzzy sets (PFS), which changed the constraining relation to $M^2 + N^2 \leq 1$.

Prior to that, Atanassov[8] proposed IFS of second type to deal with the same issue. In 2011, Ciucci [15] introduced the term orthopair as an alternative pair of M and N degrees. This gives rise to the generalized orthopair fuzzy sets or called q-rung orthopair fuzzy sets (q-ROFS), which satisfy $Mq + Nq \leq 1$ for any q positive integers [29]. Vassilev et al. [25] defined a similar concept called IFS of q-type to generalize the IFS. Note that this set can be reduced to IFS for q = 1, PFS for q = 2 and Fermatean fuzzy sets (FFS), which is another special form of q-ROFS with q = 3 [24]. Similarly, several studies have explored the q-ROFS in the cases of aggregation operations [21, 2], similarity measures [16,5], and some applications in decision-making problems [4, 3]. In general, the expression of q-ROFS is acknowledged to provide greater flexibility and expressive power for decision-makers in representing their preferences compared to IFS [29].In 1989, IFS was expanded from what was originally a singular point into an area in an intuitionistic fuzzy interpretation triangle (IFIT) with a rectangular shape called interval-valued IFS(IVIFS) [10]. The main motivation for this extension was to deal with imprecise of M and of N values. Recently, Atanassov introduced another extension of M and N interpretation into a circle called circular IFS (CIFS) [9]. This set is characterized by a 3-tuple containing M, N, and radius for each element. The difference with IFS lies in the existence of a circular imprecision area with radius r. Compared to IVIFS, CIFS has an equidistant centre point and boundary, which is not necessarily true for IVIFS, as their boundaries can take various shapes and distances from the centre point. The CIFS theory is still at an early stage of its development. Hence, not much research has been conducted on it. Initially, Atanassov [9] defined the basic relations and operations for CIFS with $r \in [0, 1]$, but then has been expanded to $r \in [0,\sqrt{2}]$ to cover the whole region in the IFIT [11].Some studies on CIFS have been conducted, including distance measures [11, 14] and divergence measures for CIFS [20]. Other than that, some extensions of decision-making models under the CIFS environment have also been proposed recently, such as in technique for order preference by similarity to ideal solution (TOPSIS) [18, 6], multiple criteria optimization and compromise solution (VIKOR) [19], the integration of analytic hierarchy process (AHP) and VIKOR [23] and a general multiple criteria decision making (MCDM) model [12]. In this paper, the concept of (3,2)- fuzzy set is investigated and compared with other types of uncertainty sets. We define Pythagorean fuzzy sub algebras in BCK-algebras and BCI-algebras and study their properties. A given (3,2)- fuzzy sub algebra is used to create a new (3,2)- fuzzy sub algebra. Also we obtain the intersection of two (3,2)- fuzzy sub algebras to be (3,2)- fuzzy sub algebra is proved and an example is given to show that the union of (3,2)- fuzzy sub algebras may not be (3,2)- fuzzy sub algebra. The characterization of cut set is also used in (3,2)- fuzzy sub algebra. The homomorphic image and pre image of (3,2)- fuzzy sub algebra is discussed. It turns out that Pythagorean fuzzy sub algebra is a subclass of (3,2)- fuzzy sub algebra.

2. Preliminaries

In this section, we just recall the basic concepts of BCK/BCI algebras.

If a set X has a special element '0' and binary operation '$*$' satisfying the conditions:

$$(BCI1): (\forall x, y, z \in X)\left(\left((x * y) * (x * z)\right) * (z * y) = 0\right),$$

$$(BCI2): (\forall x, y, z \in X)\left(\left(x * (x * y)\right) * y = 0\right),$$

$$(BCI3): (\forall x \in X)\ (x * x = 0),$$

$$(BCI4): (\forall x, y \in X)(x * y = 0\,, y * x = 0\ \Rightarrow x = y).$$

Then we say that X is BCI-algebra.

If the BCI-algebra X satisfies the following identity:

$(BCI5): (\forall x \in X)\ (0 * x = 0)$, then X is BCK-algebra.

The order relation "$\leq$" in a BCK/BCI-algebra X is defined as follows:

$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0)$……………………………….(1)

Every BCK/BCI-algebra X satisfies the following conditions:

$(\forall x \in X)\ (0 * x = 0)$,……………………………………………(2)

$(\forall x, y, z \in X)(x \leq y \Rightarrow x * z \leq y * z\,, z * y \leq z * x)$……………..(3)

$(\forall x, y, z \in X)\left((x * y) * z = (x * z) * y\right)$…………………………(4)

A non-empty subset 'A' of a BCK/BCI-algebra X is called sub algebra of X if $x * y \in A$ for all $x, y \in A$.

Every ideal 'S' of a BCK/BCI-algebra X satisfies the next assertion.

$(\forall x, y \in X)(x \leq y\,,\ y \in S \Rightarrow x \in S)$……………………….…….(5)

Let X and Y be BCK/BCI-algebras. A mapping $\varphi: X \to Y$ is called a homomorphism if it satisfies: $(\forall x, y \in X)\left(\varphi(x * y) = \varphi(x) * \varphi(y)\right)$…..……………(6)

Let $\mu_A: X \to [0,1]$ and $\vartheta_A: X \to [0,1]$ be fuzzy sets in a set X. The structure $\Delta = \{(x, \mu_A(x), \vartheta_A(x))/x \in X\}$ is called

(i) an intuitionistic fuzzy set in X, if it satisfies

$(\forall x \in X)\ (0 \leq \mu_A(x) + \vartheta_A(x) \leq 1)$…………………..(7)

(ii) a Pythagorean fuzzy set in X, if it satisfies

$(\forall x \in X)\ \left(0 \leq {\mu_A}^2(x) + {\vartheta_A}^2(x) \leq 1\right)$………………..(8)

(iii) a (3,2)-fuzzy set in X, if it satisfies

$(\forall x \in X)\left(0 \leq \mu_A{}^3(x) + \vartheta_A{}^2(x) \leq 1\right)$……………….(9)

(iv) a square root (SR) fuzzy set in X, if it satisfies

$(\forall x \in X)\left(0 \leq \mu_A{}^2(x) + \sqrt{\vartheta_A(x)} \leq 1\right)$……………….(10)

(v) a cube root (CR) fuzzy set in X, if it satisfies

$(\forall x \in X)\left(0 \leq \mu_A{}^3(x) + \sqrt[3]{\vartheta_A(x)} \leq 1\right)$……………….(11)

3. (3,2)- Fuzzy set

Definition-3.1: Let $\mu_\Delta: X \to [0,1]$ and $\vartheta_\Delta: X \to [0,1]$ be fuzzy sets in a set X. Let $(\forall x \in X)\left(0 \leq \mu_\Delta{}^3(x) + \vartheta_\Delta{}^2(x) \leq 1\right)$, then the structure $P = \{\langle x, \mu_\Delta(x), \vartheta_\Delta(x)\rangle / x \in X\}$ is called the (3,2)- fuzzy set in X.

In what follows, we apply the notations $\mu_\Delta{}^3(x)$ and $\vartheta_\Delta{}^2(x)$ instead of $\left(\mu_\Delta(x)\right)^3$ and $(\vartheta_\Delta(x))^2$, respectively and the (3,2)- Fuzzy set on X and is simply denoted by $P = (X, \mu_\Delta, \vartheta_\Delta)$. The collection of (3,2)- fuzzy sets on X is denoted by $F_3{}^2(X)$.

Example 3.2: Let $X = \{0, l, m, n, r\}$ be the set and define fuzzy sets $\mu_\Delta: X \to [0,1]$ and $\vartheta_\Delta: X \to [0,1]$ as follows:

X	0	l	m	n	r
μ_Δ	0.93	0.74	0.92	0.55	0.67
ϑ_Δ	0.87	0.41	0.74	0.65	0.57

Then $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a $(5,3)$-fuzzy set on X if $3 \geq 9$. But it is not a $(5,3)$-fuzzy set on X for $3 \leq 8$ because $(0.93)^5 + (0.87)^8 = 1.0239 > 1$.

Example 3.3: Consider the $(5,3)$-fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$ on X for $3 \geq 9$ in previous example. It is not intuitionistic fuzzy set because of $\mu_\Delta(0) + \vartheta_\Delta(0) = 0.93 + 0.87 = 1.8 > 1$.

Since $\mu_A{}^2(m) + \vartheta_A{}^2(m) = (0.92)^2 + (0.74)^2 = 1.394 > 1$, we know that $P = (X, \mu_\Delta, \vartheta_\Delta)$ is not a Pythagorean fuzzy set on X.

Since $\mu_A{}^3(m) + \vartheta_A{}^2(m) = (0.92)^3 + (0.74)^2 = 1.3263 > 1$, we know that $P = (X, \mu_\Delta, \vartheta_\Delta)$ is not a (3, 2)-fuzzy set on X.

Because of $\mu_A{}^3(0) + \vartheta_A{}^3(0) = (0.93)^3 + (0.87)^3 = 1.4629 > 1$, we know that $P = (X, \mu_\Delta, \vartheta_\Delta)$ is not a fermatean fuzzy set on X.

Finally $P = (X, \mu_\Delta, \vartheta_\Delta)$ is not a 5-pythogorean fuzzy set on X, since $\mu_A{}^5(0) + \vartheta_A{}^5(0) = (0.93)^5 + (0.87)^5 = 1.194 > 1$.

Definition 3.4: We define a binary relation '$\lesssim$' and the equality '=' in $F_3{}^2(X)$ as follows

$P_1 \lesssim P_2 \Leftrightarrow \mu_{\Delta_1} \leq \mu_{\Delta_2}, \vartheta_{\Delta_1} \geq \vartheta_{\Delta_2}$………………………………(1)

$P_1 = P_2 \Leftrightarrow \mu_{\Delta_1} = \mu_{\Delta_2}, \vartheta_{\Delta_1} = \vartheta_{\Delta_2}$....................................(2)

for all $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$, $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2})$ and $P_1 \neq P_2$. It is clear that $({F_3}^2(X), \precsim)$ is a partially ordered set.

Definition 3.5: For all $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in {F_3}^2(X)$, we define the union and the intersection as follows:

$P_1 \cup P_2 = (X, \mu_{\Delta_1} \cup \mu_{\Delta_2}, \vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})$........................(3)

$P_1 \cap P_2 = (X, \mu_{\Delta_1} \cap \mu_{\Delta_2}, \vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})$........................(4)

Where

$$\mu_{\Delta_1} \cup \mu_{\Delta_2} : X \to [0,1], x \to \max\{\mu_{\Delta_1}(x), \mu_{\Delta_2}(x)\}$$

$$\mu_{\Delta_1} \cap \mu_{\Delta_2} : X \to [0,1], x \to \min\{\mu_{\Delta_1}(x), \mu_{\Delta_2}(x)\}$$

$$\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2} : X \to [0,1], x \to \min\{\vartheta_{\Delta_1}(x), \vartheta_{\Delta_2}(x)\}$$

$$\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2} : X \to [0,1], x \to \max\{\vartheta_{\Delta_1}(x), \vartheta_{\Delta_2}(x)\}$$

It is clear that the union and intersection are associative binary operators in ${F_3}^2(X)$.

Example 3.6: Let $X = \{0, l, m, n\}$ be a set and define (3, 2)-fuzzy sets $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2})$ on X by the tables below:

X	0	l	m	n
$\mu_{\Delta_1}(x)$	0.93	0.74	0.82	0.55
$\vartheta_{\Delta_1}(x)$	0.17	0.43	0.19	0.66

and

X	0	l	m	n
$\mu_{\Delta_2}(x)$	0.85	0.84	0.69	0.75
$\vartheta_{\Delta_2}(x)$	0.37	0.25	0.48	0.36

respectively. Then $P_1 \cup P_2$ of P_1 and P_2 is given below

X	0	l	m	n
$(\mu_{\Delta_1} \cup \mu_{\Delta_2})(x)$	0.93	0.84	0.82	0.75
$(\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})(x)$	0.17	0.25	0.19	0.36

Also, the intersection $P_1 \cap P_2$ of P_1 and P_2 is given by the table below

X	0	l	m	n
$(\mu_{\Delta_1} \cap \mu_{\Delta_2})(x)$	0.85	0.74	0.69	0.55
$(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2})(x)$	0.37	0.43	0.48	0.66

Proposition 3.7: Let $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in {F_3}^2(X)$.

Then $P_1 \cap P_2 = P_2 \cap P_1$ (Commutative Law)

$P_1 \cup P_2 = P_2 \cup P_1$ (Commutative Law)

$(P_1 \cap P_2) \cup P_3 = P_2$ (Absorption Law)

$(P_1 \cup P_2) \cap P_3 = P_2$ (Absorption Law)

Proof: Straight forward.

Proposition 3.8: Every element of ${F_3}^2(X)$ is idempotent under the binary operation '$\cup$' and '$\cap$'.

Proof: Straight forward.

Theorem 3.9: $({F_3}^2(X), \cup, C_{01})$ and $({F_3}^2(X), \cap, C_{10})$ are commutative monoids where $C_{01} = (X, \tilde{0}, \tilde{1})$ and $C_{10} = (X, \tilde{1}, \tilde{0})$ with $\tilde{0}: X \to [0,1], x \mapsto \tilde{0}$ and $\tilde{1}: X \to [0,1], x \mapsto \tilde{1}$.

Proof: The proof is obvious.

Definition 3.10: The complement of $P = (X, \mu_\Delta, \vartheta_\Delta) \in {F_3}^2(X)$ is denoted by $P^C = (X, {\mu_\Delta}^C, {\vartheta_\Delta}^C)$ and is defined to be also (3,2)- fuzzy set $P^C = (X, \vartheta_\Delta, \mu_\Delta)$.

Example 3.11: Consider a (3,2)- fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$ and $X = \{0, 1, 2, 3\}$ which is defined by the following table

X	0	1	2	3
$\mu_\Delta(x)$	0.76	0.34	0.85	0.47
$\vartheta_\Delta(x)$	0.57	0.53	0.39	0.67

Then its complement $P^C = (X, {\mu_\Delta}^C, {\vartheta_\Delta}^C)$ is given as follows

X	0	1	2	3
${\mu_\Delta}^C(x)$	0.57	0.53	0.39	0.67
${\vartheta_\Delta}^C(x)$	0.76	0.34	0.85	0.47

Proposition 3.12: If $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in {F_3}^2(X)$, then $(P_1 \cap P_2)^C = {P_1}^C \cup {P_2}^C$ and $(P_1 \cup P_2)^C = {P_1}^C \cap {P_2}^C$.

Proof: For a given $P_1 = (X, \mu_{\Delta_1}, \vartheta_{\Delta_1})$ and $P_2 = (X, \mu_{\Delta_2}, \vartheta_{\Delta_2}) \in {F_3}^2(X)$, we have

$$(P_1 \cap P_2)^C = \left(X, (\mu_{\Delta_1} \cap \mu_{\Delta_2})^C, (\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2})^C\right)$$

$$= \left(X, \vartheta_{\Delta_1} \cup \vartheta_{\Delta_2}, \mu_{\Delta_1} \cap \mu_{\Delta_2}\right)$$
$$= \left(X, \vartheta_{\Delta_1}, \mu_{\Delta_1}\right) \cup \left(X, \vartheta_{\Delta_2}, \mu_{\Delta_2}\right)$$
$$= \left(X, {\mu_{\Delta_1}}^C, {\vartheta_{\Delta_1}}^C\right) \cup \left(X, {\mu_{\Delta_2}}^C, {\vartheta_{\Delta_2}}^C\right)$$
$$= {P_1}^C \cup {P_2}^C$$

The same way induces $(P_1 \cup P_2)^C = {P_1}^C \cap {P_2}^C$.

4. (3,2)- fuzzy sub algebras of BCK/BCI-algebras

In what follows, let X represent the BCK-algebra or BCI-algebra unless otherwise specified.

Definition-4.1: A (3,2)- fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$is called a (3,2)- fuzzy sub algebra of X if it satisfies

$(\forall x, y \in X)\ {\mu_\Delta}^3(x * y) \geq \min\{{\mu_\Delta}^3(x), {\mu_\Delta}^3(y)\}$

$${\vartheta_\Delta}^2(x * y) \leq \max\{{\vartheta_\Delta}^2(x), {\vartheta_\Delta}^2(y)\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Example 4.2: Let $X = \{0, 1, 2, 3\}$ be the set with binary operation '$*$' in the table

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Lemma 4.3: Every (3,2)- fuzzy sub algebra $P = (X, \mu_\Delta, \vartheta_\Delta)$ satisfies:

$(\forall x \in X)\left({\mu_\Delta}^3(0) \geq {\mu_\Delta}^3(x), {\vartheta_\Delta}^2(0) \leq {\vartheta_\Delta}^2(x)\right)$.

Proof: From the definition-A, we have the following

${\mu_\Delta}^3(0) = {\mu_\Delta}^3(x * x) \geq \min\{{\mu_\Delta}^3(x), {\mu_\Delta}^3(x)\} = {\mu_\Delta}^3(x)$,

${\vartheta_\Delta}^2(0) = {\vartheta_\Delta}^2(x * x) \leq \max\{{\vartheta_\Delta}^2(x), {\vartheta_\Delta}^2(x)\} = {\vartheta_\Delta}^2(x)$ for all $x \in X$.

Theorem 4.4: If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a (3,2)- fuzzy sub algebra of X, then the set $X_P = \{x \in X / {\mu_\Delta}^3(x) = {\mu_\Delta}^3(0), {\vartheta_\Delta}^2(x) = {\vartheta_\Delta}^2(0)\}$ is a sub algebra of X.

Proof: If $x, y \in X_P$, then ${\mu_\Delta}^3(x) = {\mu_\Delta}^3(0)$, ${\mu_\Delta}^3(y) = {\mu_\Delta}^3(0)$, ${\vartheta_\Delta}^2(x) = {\vartheta_\Delta}^2(0)$, ${\vartheta_\Delta}^2(y) = {\vartheta_\Delta}^2(0)$. It follows from the definition-A that

${\mu_\Delta}^3(x * y) \geq \min\{{\mu_\Delta}^3(x), {\mu_\Delta}^3(y)\} = {\mu_\Delta}^3(0)$ and

${\vartheta_\Delta}^2(x * y) \leq \max\{{\vartheta_\Delta}^2(x), {\vartheta_\Delta}^2(y)\} = {\vartheta_\Delta}^2(0)$.

By combining this and previous lemma we derive $\mu_\Delta{}^3(x*y) = \mu_\Delta{}^3(0)$ and $\vartheta_\Delta{}^2(x*y) = \vartheta_\Delta{}^2(0)$, and so $x*y \in X_P$. Hence X_P is a sub algebra of X. Hence the proof.

Given a (3,2)- fuzzy set $P = (X, \mu_\Delta, \vartheta_\Delta)$ and define a new (3,2)- fuzzy set $P^* = (X, \mu_\Delta{}^*, \vartheta_\Delta{}^*)$ on X as follows $\mu_\Delta{}^*: X \to [0,1],\ x \mapsto \frac{\mu_\Delta(x)}{\sup\{\mu_\Delta(x)/x\in X\}}$,

$\vartheta_\Delta{}^*: X \to [0,1],\ x \mapsto \frac{\vartheta_\Delta(x)}{\inf\{\vartheta_\Delta(x)/x\in X\}}$, where $\inf\{\vartheta_\Delta(x)/x \in X\} \neq 0$.

Theorem 4.5: If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a (3,2)- fuzzy sub algebra of X with $\vartheta_\Delta(0) \neq 0$, then $P^* = (X, \mu_\Delta{}^*, \vartheta_\Delta{}^*)$ is a (3,2)- fuzzy sub algebra of X.

Proof: If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is a (3,2)- fuzzy sub algebra of X, then $\sup\{\mu_\Delta(x)/x \in X\} = \mu_\Delta(0)$ and $\inf\{\vartheta_\Delta(x)/x \in X\} = \vartheta_\Delta(0) \neq 0$.

Then we have

$$\begin{aligned}\mu_\Delta{}^3(x*y) &= \left(\frac{\mu_\Delta(x*y)}{\sup\{\mu_\Delta(x*y)/x*y \in X\}}\right)^3 = \left(\frac{\mu_\Delta(x*y)}{\mu_\Delta(0)}\right)^3 = \frac{\mu_\Delta{}^3(x*y)}{\mu_\Delta{}^3(0)}\\ &\geq \frac{1}{\mu_\Delta{}^3(0)}\min\{\mu_\Delta{}^3(x), \mu_\Delta{}^3(y)\}\\ &= \min\left\{\frac{\mu_\Delta{}^3(x)}{\mu_\Delta{}^3(0)}, \frac{\mu_\Delta{}^3(y)}{\mu_\Delta{}^3(0)}\right\} = \min\left\{\left(\frac{\mu_\Delta(x)}{\mu_\Delta(0)}\right)^3, \left(\frac{\mu_\Delta(y)}{\mu_\Delta(0)}\right)^3\right\}\\ &= \min\{\mu_\Delta{}^{*3}(x), \mu_\Delta{}^{*3}(y)\} \text{ and}\end{aligned}$$

$$\begin{aligned}\vartheta_\Delta{}^2(x*y) &= \left(\frac{\vartheta_\Delta(x*y)}{\inf\{\vartheta_\Delta(x*y)/x*y \in X\}}\right)^2 = \left(\frac{\vartheta_\Delta(x*y)}{\vartheta_\Delta(0)}\right)^2 = \frac{\vartheta_\Delta{}^2(x*y)}{\vartheta_\Delta{}^2(0)}\\ &\leq \frac{1}{\vartheta_\Delta{}^2(0)}\max\{\vartheta_\Delta{}^2(x), \vartheta_\Delta{}^2(y)\}\\ &= \max\left\{\frac{\vartheta_\Delta{}^2(x)}{\vartheta_\Delta{}^2(0)}, \frac{\vartheta_\Delta{}^2(y)}{\vartheta_\Delta{}^2(0)}\right\} = \min\left\{\left(\frac{\vartheta_\Delta(x)}{\vartheta_\Delta(0)}\right)^2, \left(\frac{\vartheta_\Delta(y)}{\vartheta_\Delta(0)}\right)^2\right\}\\ &= \max\{\vartheta_\Delta{}^{*2}(x), \vartheta_\Delta{}^{*2}(y)\}\end{aligned}$$

for all $x, y \in X$. So $P^* = (X, \mu_\Delta{}^*, \vartheta_\Delta{}^*)$ is a (3,2)- fuzzy sub algebra of X.

Theorem 4.6: If P_1 and P_2 are (3,2)- fuzzy sub algebras of X, then their intersection $P_1 \cap P_2$ is also a (3,2)- fuzzy algebra of X.

Proof: For every $x \in X$, we have

$$\begin{aligned}(\mu_{\Delta_1} \cap \mu_{\Delta_2})^3(x*y) &= \min\{\mu_{\Delta_1}{}^3(x*y), \mu_{\Delta_2}{}^3(x*y)\}\\ &\geq \min\left\{\min\{\mu_{\Delta_1}{}^3(x), \mu_{\Delta_1}{}^3(y)\}, \min\{\mu_{\Delta_2}{}^3(x), \mu_{\Delta_2}{}^3(y)\}\right\}\\ &= \min\left\{\min\{\mu_{\Delta_1}{}^3(x), \mu_{\Delta_2}{}^3(x)\}, \min\{\mu_{\Delta_1}{}^3(y), \mu_{\Delta_2}{}^3(y)\}\right\}\\ &= \min\left\{(\mu_{\Delta_1} \cap \mu_{\Delta_2})^3(x), (\mu_{\Delta_1} \cap \mu_{\Delta_2})^3(y)\right\} \text{ and}\end{aligned}$$

$$\left(\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2}\right)^2(x*y) = \max\{\vartheta_{\Delta_1}{}^2(x*y), \vartheta_{\Delta_2}{}^2(x*y)\}$$
$$\leq \max\left\{\max\{\vartheta_{\Delta_1}{}^2(x), \vartheta_{\Delta_1}{}^2(y)\}, \max\{\vartheta_{\Delta_2}{}^2(x), \vartheta_{\Delta_2}{}^2(y)\}\right\}$$
$$= \max\left\{\max\{\vartheta_{\Delta_1}{}^2(x), \vartheta_{\Delta_2}{}^2(x)\}, \max\{\vartheta_{\Delta_1}{}^2(y), \vartheta_{\Delta_2}{}^2(y)\}\right\}$$
$$= \max\left\{\left(\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2}\right)^2(x), \left(\vartheta_{\Delta_1} \cup \vartheta_{\Delta_2}\right)^2(y)\right\}$$

Then, $P_1 \cap P_2$ is a (3,2)- fuzzy sub algebra of X.

Example 4.7: Let $X = \{0, 1, 2, 3\}$ be the set with binary operation '$*$' in the following table

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Note: $1 * 2 = 2 * 1 = 3.$

$3 * 2 = 2 * 3 = 1.$

Thus, X is BCI-algebra.

Let's define $P_1 = \left(X, \mu_{\Delta_1}, \vartheta_{\Delta_1}\right)$, $P_2 = \left(X, \mu_{\Delta_2}, \vartheta_{\Delta_2}\right) \in F_3{}^2(X)$ in the following table below, respectively

X	0	1	2	3
$\mu_{\Delta_1}(x)$	0.73	0.61	0.49	0.48
$\vartheta_{\Delta_1}(x)$	0.21	0.53	0.30	0.52

and

X	0	1	2	3
$\mu_{\Delta_2}(x)$	0.73	0.51	0.53	0.52
$\vartheta_{\Delta_2}(x)$	0.24	0.56	0.63	0.63

Then P_1 and P_2 are (3,2)- fuzzy sub algebras of X.

Then union $P_1 \cup P_2$ is calculated as follows.

X	0	1	2	3
$\left(\mu_{\Delta_1} \cup \mu_{\Delta_2}\right)(x)$	0.73	0.61	0.53	0.52
$\left(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2}\right)(x)$	0.21	0.53	0.30	0.52

and it is not (3,2)- fuzzy sub algebra of X. Because of

$$\left(\mu_{\Delta_1} \cup \mu_{\Delta_2}\right)^3(3*1) = \left(\mu_{\Delta_1} \cup \mu_{\Delta_2}\right)^3(2)$$

$$\left(\mu_{\Delta_1} \cup \mu_{\Delta_2}\right)^3(2) = \max\{\mu_{\Delta_1}{}^3(2), \mu_{\Delta_2}{}^3(2)\}$$

$= \max\{(0.49)^3, (0.53)^3\}$

$(0.53)^3 \ngeq (0.61)^3$

$= \min\left\{\left(\mu_{\Delta_1} \cup \mu_{\Delta_2}\right)^3(3), \left(\mu_{\Delta_1} \cup \mu_{\Delta_2}\right)^3(1)\right\}$ and

$\left(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2}\right)^2(2 * 1) = \left(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2}\right)^2(3)$

$\left(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2}\right)^2(3) \quad = \min\{\vartheta_{\Delta_1}{}^2(3), \vartheta_{\Delta_2}{}^2(3)\}$

$= \min\{(0.52)^3, (0.63)^3\}$

$(0.52)^2 \nleq (0.30)^2$

$= \max\left\{\left(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2}\right)^2(2), \left(\vartheta_{\Delta_1} \cap \vartheta_{\Delta_2}\right)^2(1)\right\}$

Definition 4.8: Let $P = (X, \mu_\Delta, \vartheta_\Delta) \in F_3{}^2(X)$. For every $(\alpha, \beta) \in [0,1] \times [0,1]$ with

$0 \leq \alpha^3 + \beta^2 \leq 1,\ \ P_{(\alpha,\beta)} = P_\alpha \cap P_\beta$……………………….(1)

which is called a cut set of P when

$P_\alpha = \{x \in X / \mu_\Delta{}^3(x) \geq \alpha\}$ and $P_\beta = \left\{x \in X / \vartheta_\Delta{}^2(x) \leq \beta\right\}$.

Proposition 4.9: Let $P = (X, \mu_\Delta, \vartheta_\Delta)$, $Q = (X, \delta_\Delta, \gamma_\Delta) \in F_3{}^3(X)$. Then

$P \leq Q \Rightarrow P_{(\alpha,\beta)} \subseteq Q_{(\alpha,\beta)}$……………………………………………….(2)

$(\forall (m.n) \in [0,1] \times [0,1])\left(m \leq \alpha, n \geq \beta \Rightarrow P_{(\alpha,\beta)} \subseteq P_{(m,n)}\right)$.

Proof: Assume that $P \leq Q$ and let $x \in P_{(\alpha,\beta)}$. Then $\mu_\Delta \leq \delta_\Delta$ and $\vartheta_\Delta \geq \gamma_\Delta$.

That is $\mu_\Delta(x) \leq \delta_\Delta(x)$ and $\vartheta_\Delta(x) \geq \gamma_\Delta(x)$ for all $x \in X$.

It follows that $\alpha \leq \mu_\Delta{}^3(x) \leq \delta_\Delta{}^3(x)$ and $\beta \geq \vartheta_\Delta{}^2(x) \geq \gamma_\Delta{}^2(x)$. Thus $x \in Q_{(\alpha,\beta)}$.

Hence the proof.

Now let $(m.n) \in [0,1] \times [0,1]$ be such that $m \leq \alpha, n \geq \beta$.

If $x \in P_{(\alpha,\beta)}$, then $\mu_\Delta{}^3(x) \geq \alpha \geq m$ and $\vartheta_\Delta{}^2(x) \leq \beta \leq n$. Then $x \in P_{(m,n)}$.

So $P_{(\alpha,\beta)} \subseteq P_{(m,n)}$.

Theorem 4.10: If $P = (X, \mu_\Delta, \vartheta_\Delta)$ is (3,2)- fuzzy sub algebra of X, then it's cut set $P_{(\alpha,\beta)}$ is a sub algebra of X.

Proof: Let $x, y \in P_{(\alpha,\beta)}$. Then $\mu_\Delta{}^3(x) \geq \alpha$ and $\mu_\Delta{}^3(y) \geq \alpha$, $\vartheta_\Delta{}^2(x) \leq \beta$ and $\vartheta_\Delta{}^2(y) \leq \beta$.

It follows from (1) that

$\mu_\Delta{}^3(x * y) \geq \min\{\mu_\Delta{}^3(x), \mu_\Delta{}^3(y)\} \geq \min\{\alpha, \alpha\} \geq \alpha$ and

$\vartheta_\Delta{}^2(x * y) \leq \max\left\{\vartheta_\Delta{}^2(x), \vartheta_\Delta{}^2(y)\right\} \leq \max\{\beta, \beta\} \leq \beta$.

Thus $x * y \in P_{(\alpha,\beta)}$. So $P_{(\alpha,\beta)}$ is a sub algebra of X.

Theorem 4.11: For a given P, if it's cut set $P_{(\alpha,\beta)}$ is a sub algebra of X for every

$(\alpha,\beta) \in [0,1]\times[0,1]$ with $0 \le \alpha^3+\beta^2 \le 1$, then $P=(X,\mu_\Delta,\vartheta_\Delta)$ is (3,2)- fuzzy sub algebra of X.

Proof: Assume that $P_{(\alpha,\beta)}$ is a sub algebra of X for every $(\alpha,\beta) \in [0,1]\times[0,1]$ with $0 \le \alpha^3+\beta^2 \le 1$. For every $x, y \in X$, we put $\alpha_x = {\mu_\Delta}^3(x)$, $\beta_x = {\vartheta_\Delta}^2(x)$ and $\alpha_y = {\mu_\Delta}^3(y)$, $\beta_y = {\vartheta_\Delta}^2(y)$.

Then $x, y \in P_{(\alpha,\beta)}$ for $\alpha = \min\{\alpha_x, \alpha_y\}$ and $\beta = \max\{\beta_x, \beta_y\}$. Thus $x * y \in P_{(\alpha,\beta)}$.

It follows that

${\mu_\Delta}^3(x*y) \ge \alpha = \min\{\alpha_x,\alpha_y\} = \min\{{\mu_\Delta}^3(x), {\mu_\Delta}^3(y)\}$ and

${\vartheta_\Delta}^2(x*y) \le \beta = \max\{\beta_x,\beta_y\} = \max\{{\vartheta_\Delta}^2(x), {\vartheta_\Delta}^2(y)\}$.

So $P=(X,\mu_\Delta,\vartheta_\Delta)$ is (3,2)- fuzzy sub algebra of X.

Definition 4.12: Let P and Q be fermatean fuzzy sets on X and Y repectively. Let $\varphi: X \to Y$ be a mapping from a set X to a set Y.

(i) The pre image of $Q=(Y,\delta_\Delta,\gamma_\Delta)$ under φ is defined to be (3,2)- fuzzy set $\varphi^{-1}(Q)$ on X where

$\varphi^{-1}(\delta_\Delta): X \to [0,1]$, $x \to \delta_\Delta(\varphi(x))$ and $\varphi^{-1}(\gamma_\Delta): X \to [0,1]$, $x \to \gamma_\Delta(\varphi(x))$.

(ii) The image of $P=(X,\mu_\Delta,\vartheta_\Delta)$ under φ is defined to be (3,2)- fuzzy set where

$$\varphi(\mu_\Delta): Y \to [0,1] \text{ , } y \to \begin{cases} \sup\limits_{x\in\varphi^{-1}(y)} \mu_\Delta(x), & \text{if } \varphi^{-1}(y) \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$$\varphi(\vartheta_\Delta): Y \to [0,1] \text{ , } y \to \begin{cases} \inf\limits_{x\in\varphi^{-1}(y)} \vartheta_\Delta(x), & \text{if } \varphi^{-1}(y) \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 4.13: Let $\varphi: X \to Y$ be a homomorphism of BCK/BCI-algebras. If Q is (3,2)- fuzzy sub algebra of Y, then its pre image $\varphi^{-1}(Q)$ under φ is (3,2)- fuzzy sub algebra of X.

Proof: For every $x, y \in X$, we have

$$\begin{aligned}
\varphi^{-1}(\delta_\Delta)^3(x*y) &= \left(\varphi^{-1}(\delta_\Delta)(x*y)\right)^3 \\
&= \left(\delta_\Delta\left(\varphi(x*y)\right)\right)^3 = \left(\delta_\Delta\left(\varphi(x)*\varphi(y)\right)\right)^3 \\
&= {\delta_\Delta}^3\left(\varphi(x)*\varphi(y)\right) \\
&\ge \min\{{\delta_\Delta}^3(\varphi(x)), {\delta_\Delta}^3(\varphi(y))\} \\
&= \min\left\{\left(\delta_\Delta\varphi(x)\right)^3, \left(\delta_\Delta\varphi(y)\right)^3\right\} \\
&= \min\left\{\left(\varphi^{-1}(\delta_\Delta)(x)\right)^3, \left(\varphi^{-1}(\delta_\Delta)(y)\right)^3\right\} \\
&= \min\{\varphi^{-1}(\delta_\Delta)^3(x), \varphi^{-1}(\delta_\Delta)^3(y)\} \text{ and}
\end{aligned}$$

$$\varphi^{-1}(\gamma_\Delta)^2(x*y) = \left(\varphi^{-1}(\gamma_\Delta)(x*y)\right)^2$$
$$= \left(\gamma_\Delta(\varphi(x*y))\right)^2 = \left(\gamma_\Delta(\varphi(x)*\varphi(y))\right)^2$$
$$= {\gamma_\Delta}^2(\varphi(x)*\varphi(y))$$
$$\leq \max\{{\gamma_\Delta}^2(\varphi(x)), {\gamma_\Delta}^2(\varphi(y))\}$$
$$= \max\left\{(\gamma_\Delta\varphi(x))^2, (\gamma_\Delta\varphi(y))^2\right\}$$
$$= \max\left\{(\varphi^{-1}(\gamma_\Delta)(x))^2, (\varphi^{-1}(\gamma_\Delta)(y))^2\right\}$$
$$= \max\{\varphi^{-1}(\gamma_\Delta)^2(x), \varphi^{-1}(\gamma_\Delta)^2(y)\}$$

Then $\varphi^{-1}(Q)$ is a (3,2)- fuzzy sub algebra of X.

Theorem 4.14: Let $\varphi: X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If P is a (3,2)- fuzzy sub algebra of X, then the image $\varphi(P)$ under φ is (3,2)- fuzzy sub algebra of Y.

Proof: For every $y_1, y_2 \in Y$, we have

$\{x_1, x_2 \in X / x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\} \subseteq \{x \in X / x \in \varphi^{-1}(y_1 * y_2)\}$.

Then we get

$$\varphi(\mu_\Delta)^3(y_1 * y_2) = \left(\varphi(\mu_\Delta)(y_1 * y_2)\right)^3$$
$$= (\sup\{\mu_\Delta(x)/x \in \varphi^{-1}(y_1 * y_2)\})^3$$
$$\geq (\sup\{\mu_\Delta(x_1 * x_2)/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\})^3$$
$$= \min\{\sup\{{\mu_\Delta}^3(x_1)/x_1 \in \varphi^{-1}(y_1)\}, \sup\{{\mu_\Delta}^3(x_2)/x_2 \in \varphi^{-1}(y_2)\}\}$$
$$= \min\{\varphi(\mu_\Delta)^3(y_1), \varphi(\mu_\Delta)^3(y_2)\}$$

$$\varphi(\vartheta_\Delta)^2(y_1 * y_2) = \left(\varphi(\vartheta_\Delta)(y_1 * y_2)\right)^2$$
$$= (\inf\{\vartheta_\Delta(x)/x \in \varphi^{-1}(y_1 * y_2)\})^2$$
$$\leq (\inf\{\vartheta_\Delta(x_1 * x_2)/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\})^2$$
$$= \inf\{{\vartheta_\Delta}^2(x_1 * x_2)/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\}$$
$$= \inf\{\max\{{\vartheta_\Delta}^2(x_1), {\vartheta_\Delta}^2(x_2)\}/x_1 \in \varphi^{-1}(y_1), x_2 \in \varphi^{-1}(y_2)\}$$
$$= \max\left\{\inf\{{\vartheta_\Delta}^2(x_1)/x_1 \in \varphi^{-1}(y_1)\}, \inf\{{\vartheta_\Delta}^2(x_2)/x_2 \in \varphi^{-1}(y_2)\}\right\}$$
$$= \max\{\varphi(\vartheta_\Delta)^2(y_1), \varphi(\vartheta_\Delta)^2(y_2)\}$$

Thus $\varphi(P)$ is (3,2)- fuzzy sub algebra of Y.

Finally, we discuss the relationship between intuitionistic fuzzy sub algebra and (3,2)-fuzzy algebra.

Theorem 4.15: Every intuitionistic fuzzy sub algebra is (3,2)- fuzzy sub algebra.

Proof: Let $P = (X, \mu_\Delta, \vartheta_\Delta)$ be an intuitionistic fuzzy sub algebra of X. Then
$\mu_\Delta(x * y) \geq \min\{\mu_\Delta(x), \mu_\Delta(y)\}$ and $\vartheta_\Delta(x * y) \leq \max\{\vartheta_\Delta(x), \vartheta_\Delta(y)\}$ for all $x, y \in X$.
We consider the following cases:
Case(i): $\mu_\Delta(x) \geq \mu_\Delta(y)$ and $\vartheta_\Delta(x) \geq \vartheta_\Delta(y)$,
Case(ii): $\mu_\Delta(x) \geq \mu_\Delta(y)$ and $\vartheta_\Delta(x) < \vartheta_\Delta(y)$,
Case(iii): $\mu_\Delta(x) < \mu_\Delta(y)$ and $\vartheta_\Delta(x) \geq \vartheta_\Delta(y)$,
Case(iv): $\mu_\Delta(x) < \mu_\Delta(y)$ and $\vartheta_\Delta(x) < \vartheta_\Delta(y)$.
Case(i) implies $\mu_\Delta{}^3(x) \geq \mu_\Delta{}^3(y)$ and $\vartheta_\Delta{}^2(x) \geq \vartheta_\Delta{}^2(y)$, then

$$\begin{aligned}\mu_\Delta{}^3(x * y) &= \left(\mu_\Delta(x * y)\right)^3 \\ &\geq (\min\{\mu_\Delta(x), \mu_\Delta(y)\})^3 \\ &= \min\{\mu_\Delta{}^3(x), \mu_\Delta{}^3(y)\}\end{aligned}$$

$$\begin{aligned}\vartheta_\Delta{}^2(x * y) &= \left(\vartheta_\Delta(x * y)\right)^2 \\ &\leq (\max\{\vartheta_\Delta(x), \vartheta_\Delta(y)\})^2 \\ &= \max\left\{\vartheta_\Delta{}^2(x), \vartheta_\Delta{}^2(y)\right\}, \text{ for all } x, y \in X.\end{aligned}$$

In the rest of cases, the condition equation (1) can be derived in the same way. Thus, $P = (X, \mu_\Delta, \vartheta_\Delta)$ is (3,2)- fuzzy sub algebra of X.

The converse of above theorem may not be true. In fact, (3,2)- fuzzy sub algebra $P = (X, \mu_\Delta, \vartheta_\Delta)$ of X for all $(3,3) \in N \times N$ with $(3,3) \notin \{(1,1), (1,2), (2,1)\}$ in example (1) is not an intuitionistic fuzzy sub algebra of X because of

$$\mu_\Delta(3) + \mu_\Delta(3) = 0.52 + 0.63 = 1.15 > 1.$$

Conclusion: As per the sub class of intuitionistic fuzzy set, Pythagorean fuzzy set and fermatean fuzzy set, we introduced the notion of (3,2)- fuzzy set and applied it to BCK/BCI-algebras. We gave some operations for (3,2)- fuzzy set and investigated their properties. We introduce (3,2)- fuzzy sub algebra in BCK/BCI-algebras and investigated several properties. We proved that the intersection of two (3,2)- fuzzy sub algebra is also a fermatean fuzzy sub algebra and provided an example is given to the union of two (3,2)- fuzzy sub algebras may not be a (3,2)- fuzzy sub algebra.

Also, we used the cut set to obtain the structures of (3,2)- fuzzy sub algebra. We show that intuitionistic fuzzy sub algebra is a sub class of (3,2)- fuzzy sub algebra and consider the homomorphic image and pre image of (3,2)- fuzzy sub algebra.

Future work: The idea of this paper and the results obtained will be used for the study of various types of logical algebra in the future. And considering research on soft set theory and rough set theory etc. based on Pythagorean fuzzy set is also a subject of future research. It also attempts to explore the role of source in solving problems that includes uncertainty.

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