DISCRETE AND INDISCRETE (4, 2)-FUZZY TOPOLOGICAL STRUCTURES

[1]S.V. Manemaran & [2]R. Nagarajan

[1]*Professor, Department of Mathematics, Bharath Institute of Science and Technology, Bharath Institute of Higher Education and Research, Chennai, Tamilnadu, India.*

[2]*Professor, Department of Mathematics, J.J. College of Engineering and Technology, Tiruchirappalli, Tamilnadu, India.*

Abstract: In this paper, we study the concept of (4,2)-fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of (4,2)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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1.Introduction: In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs or does not belong to the set. As an extension, fuzzy set theory (See [22]) permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0; 1]. As a generalization of fuzzy set, Atanassov [1] created intuitionistic fuzzy set. Intuitionistic fuzzy set is widely used in all _elds (See [4, 5, 12, 18] for applications in algebraic structures). In 2013, Yager [19, 20, 21] introduced Pythagorean fuzzy set and compared it with intuitionistic fuzzy set. Pythagorean fuzzy set is a new extension of intuitionistic fuzzy set that conducts to simulate the vagueness originated by the real case that might arise in the sum of membership and non-membership is bigger than 1. Pythagorean fuzzy set is applied to groups (See [2]), UP-algebras (See [15]) and topological spaces (See [14]). Senapati et al. [16] introduced Fermatean fuzzy set which is another extension of intuitionistic fuzzy sets and it is applied to groups (See [17]). Ibrahim et al. [9] introduced (3; 2)-fuzzy sets and applied it to topological spaces. In this paper, we study the concept of (4,2)-fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of (4,2)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

2. Preliminaries and Various Basic Concept of BCC-algebras(BCK-algebras)

In this section, we first review some definitions and properties which will be used in the sequel.

A non-empty set G with a constant 0 and binary operation $*$ is called a BCC-algebra if it satisfies the following conditions:

a) $\left(\left((x * y) * (z * y)\right) * (x * y) = 0\right)$

b) $x * x = 0$

c) $0 * x = 0$

d) $x * 0 = 0$

e) $x * y = 0\,, y * x = 0 \;\Rightarrow x = y$

for all $x, y, z \in G$. In BCC-algebra, the following equality holds $(x * y) * x = 0$.

Obviously, any BCK-algebra is BCC-algebra but there exist BCC-algebras which are not necessarily BCK-algebra. We note that a BCC-algebra is BCK-algebra if and if only, it satisfies the equality $(x * y) * z = (x * z) * y$.

A non-empty subset ‘S’ of a BCK-algebra ‘G’ is called a sub algebra of G if it is closed under the BCC-operation. Such algebra contains the constant 0 and it is clearly a BCC-algebra, but some sub algebras may be also BCK-algebras. Moreover, there exit BCC-algebras which all sub algebras are BCK-algebras.

A mapping $\varphi: G_1 \to G_2$ of BCC-algebras is called a homomorphism if $\varphi(x * y) = \varphi(x) * \varphi(y)$ holds, for all $x, y \in G_1$.

For a non-empty given set G, let I be the closed unit interval [0, 1]. Then, an (4,2)-fuzzy set is an object of the form $A = \{\langle x, {\delta_A}^4(x), {\lambda_A}^2(x)\rangle / \, x \in G\}$, when the mappings ${\delta_A}^4: G \to I$ and ${\lambda_A}^2: G \to I$ denote the degree of membership (namely, $\delta_A(x)$) and the degree of non-membership (namely, $\lambda_A(x)$) of each element $x \in G$ to the object ‘A’ respectively satisfying $0 \le {\delta_A}^m(x) + {\lambda_A}^n(x) \le 1$ for all $x \in G$.

The complement of the (4,2)-fuzzy set is $A^C = \{\langle x, {\lambda_A}^2(x), {\delta_A}^4(x)\rangle / \, x \in G\}$. Obviously, every fuzzy A on a non-empty G is an (4, 2)-fuzzy set of the form $A = \{\langle x, {\delta_A}^4(x), 1 - {\lambda_A}^2(x)\rangle / \, x \in G\}$. For the sake of simplicity, we just write $A = \langle {\delta_A}^4, {\lambda_A}^4\rangle$ instead of $A = \{\langle x, \delta_A(x), \lambda_A(x)\rangle / \, x \in G\}$.

The (4,2)-fuzzy sets $0\sim$ and $1\sim$ in G are defined by

$0\sim = \{\langle x, 0, 1\rangle : x \in G\}$ and $1\sim = \{\langle x, 1, 0\rangle : x \in G\}$, respectively.

If φ is a mapping which maps a set G_1 into another set G_2, then the following statement hold:

(a) If $B = \{\langle y, {\delta_B}^4(y), {\lambda_B}^2(y)\rangle / \ y \in G_2\}$ is an (4,2)-fuzzy set in G_2, then the pre image of B under φ, denoted by $\varphi^{-1}(B)$, is still an (4,2)-fuzzy set in G_1, we now write $\varphi^{-1}(B) = \{\langle x,\ \varphi^{-1}(\delta_B)(x),\ \varphi^{-1}(\lambda_B)(x)\rangle / \ x \in G_1\}$.

(b) If $A = \{\langle x, {\delta_A}^4(x), {\lambda_A}^2(x)\rangle / \ x \in G_1\}$ in an (4,2)-fuzzy set in G_1, then the image of A under φ, denoted by φ(A), is also an (4,2)-fuzzy set in G_2, which is defined by $\varphi(A) = \{\langle y, \varphi_{sup}(\delta_A)(y), \varphi_{inf}(\lambda_A)(y) \rangle : y \in G_2\}$, where

$$\varphi_{sup}(\delta_A)(y) = \begin{cases} \sup\limits_{x\in \varphi^{-1}(y)} \delta_A(x), & \text{if } \varphi^{-1}(y) \neq 0, \\ 0, & \text{else where,} \end{cases}$$

$$\varphi_{inf}(\lambda_A)(y) = \begin{cases} \inf\limits_{x\in \varphi^{-1}(y)} \lambda_A(x), & \text{if } \varphi^{-1}(y) \neq 0, \\ 0, & \text{else where,} \end{cases}$$

for each $y \in G_2$.

Proposition-2.1: Let A, $A_i (i \in I)$be (4,2)-fuzzy set in G_1 and B an (4,2)-fuzzy set in G_2. If $\varphi: G_1 \to G_2$ is a function, then the following properties hold for the function φ:

(a) If φ is surjective, then $\varphi(\varphi^{-1}(B)) = B$.

(b) $\varphi^{-1}(\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} \varphi^{-1}(A_i)$.

(c) $\varphi^{-1}(1\sim) = 1\sim$.

(d) $\varphi^{-1}(0\sim) = 0\sim$.

(e) $\varphi(1\sim) = 1\sim$, if φ is surjective

(f) $\varphi(0\sim) = 0\sim$.

Definition-2.2: An (4,2)-fuzzy topology on a non-empty set G is a family τ of (4,2)-fuzzy sets in G which satisfies the following conditions:

(i) $0\sim, 1\sim \in \tau$.

(ii) If $G_1, G_2 \in \tau$, then $G_1 \cap G_2$.

(iii) If $G_j \in \tau$ for all $j \in J$, then $\cup_{i\in I} G_i \in \tau$.

The pair (G, τ) is called an (4,2)-fuzzy topological space and any (4,2)-fuzzy set in τ is called an (4,2)-fuzzy open sets in G. The topology τ on a (4,2)-fuzzy topological space is said to be an indiscrete (4,2)-fuzzy topology if it's only element are $0\sim$ and $1\sim$. On the other hand, (4,2)-fuzzy topology τ on a space G is said to be discrete (4,2)-fuzzy topology
if the topology (4,2)-fuzzy topology τ contains all (4, 2)-fuzzy subsets of G.

If A is an (4,2)-fuzzy set in an (4,2)-fuzzy topological space (G, τ), then the induced (4,2)-fuzzy topological space on A is the family of (4,2)-fuzzy sets in A which are the intersection with A of (4,2)-fuzzy sets in G. The induced (4,2)-fuzzy topology is denoted by τ_A, and the pair (A, τ_A) is called an fuzzy subspace of (G, τ).

Let (G_1, τ_1) and (G_2, τ_2) be two (4,2)-fuzzy topological spaces and $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ a function. Then φ is said to be (4,2)-fuzzy continuous function if and only if the pre image of each (4,2)-fuzzy set in τ_2 is an (4,2)-fuzzy set in τ_1. Let (G_1, τ_1) and (G_2, τ_2) be two (4,2)-fuzzy topological spaces and $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ a function. Then φ is said to be (4,2)-fuzzy open if and only if the image of each (4,2)-fuzzy set in τ_1 is an (4,2)-fuzzy set in τ_2.

3. (4,2)-fuzzy topological sub algebras

Definition-3.1: An (4,2)-fuzzy set $A = \langle {\delta_A}^4, {\lambda_A}^2 \rangle$ in G is called (4,2)-fuzzy sub algebra of G if it satisfies the following conditions;

FS1 : ${\delta_A}^4(x * y) \geq \min\{{\delta_A}^4(x), {\delta_A}^4(y)\}$

FS2 : ${\lambda_A}^2(x * y) \leq \max\{{\lambda_A}^2(x), {\lambda_A}^2(y)\}$, for all $x, y \in G$.

Example-3.2: Let $G = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table.

+	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Let $A = \langle {\delta_A}^4, {\lambda_A}^2 \rangle$ be an (4,2)-fuzzy set in G defined by ${\delta_A}^4(4) = 0.07$, ${\delta_A}^4(x) = 0.6$, ${\lambda_A}^2(x) = 0.5$ and ${\lambda_A}^2(4) = 0.06$ for all $x \neq d$. Then A is (4,2)-fuzzy sub algebra of G.

Definition-3.3: Let τ_1 and τ_2 be an (4,2)-fuzzy topologies on BCC-algebras G_1 and G_2 respectively. A function $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ is called an (4,2)-fuzzy continuous function which maps (G_1, τ_1) and (G_2, τ_2) if φ satisfies the following conditions:

(i) For every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

(ii) For every (4,2)-fuzzy sub algebra A (of G_2) in τ_2, $\varphi^{-1}(A)$ is (4,2)-fuzzy sub algebra (of G_1) in τ_1.

Proposition-3.4: If in τ_1 is an (4,2)-fuzzy topology on a BCC-algebra G_1 and τ_2 is an (4,2)-fuzzy topology on a BCC-algebra G_2, then every function $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is a (4,2)-fuzzy continuous function.

Proof: Since τ_2 is an indiscrete (4,2)-fuzzy topology, $\tau_2 = (0\sim, 1\sim)$..

Let $\varphi: G_1 \to G_2$ be any mapping of BCC-algebras. Then, every member of τ_2 is an (4,2)-fuzzy topology on a BCC-algebra G_2.

We now show that φ is (4,2)-fuzzy continuous function. We only need to prove that for every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

For this purpose, we let $0\sim \in \tau_2$. Then for any $x \in G_1$, we have
$\varphi^{-1}(0\sim)(x) = 0\sim(\varphi(x)) = 0 = 0\sim(x)$. This show that $\left(\varphi^{-1}(0\sim)\right) = 0\sim \in \tau_1$.

On the other hand, if $1\sim \in \tau_2$ and $x \in G_1$, then
$\varphi^{-1}(1\sim)(x) = 1\sim(\varphi(x)) = 1 = 1\sim(x)$. Thus $\left(\varphi^{-1}(1\sim)\right) = 1\sim \in \tau_1$.

This show that φ is indeed an (4,2)-fuzzy continuous function of G_1 to G_2.

Theorem-3.5: Let τ_1 and $\tau_2$be any two discrete (4,2)-fuzzy topologies defined on the BCC-algebras G_1 and G_2 respectively. Then every homomorphism $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is an (4,2)-fuzzy continuous function.

Proof: Since τ_1 and τ_2 are discrete (4,2)-fuzzy topologies on the BCC-algebras G_1 and G_2 respectively, we have $\varphi^{-1}(A) \in \tau_1$ for every $A \in \tau_2$.

We note that φ is not the usual inverse homomorphism from G_2 to G_1.

Let $A = \langle {\delta_A}^4, {\lambda_A}^2 \rangle$ be an(4,2)-fuzzy sub algebra (of G_2) in τ_2. Then for $x, y \in G_1$, we have,

$$\left(\varphi^{-1}({\delta_A}^4)\right)(x * y) = {\delta_A}^4(\varphi(x * y))$$
$$= {\delta_A}^4(\varphi(x) * \varphi(y))$$
$$\geq \min\{{\delta_A}^4(\varphi(x)), {\delta_A}^4(\varphi(y))\}$$
$$= \min\left\{\left(\varphi^{-1}({\delta_A}^4)\right)(x), \left(\varphi^{-1}(4)\right)(y)\right\} \text{ and}$$

$$\left(\varphi^{-1}({\lambda_A}^2)\right)(x * y) = {\lambda_A}^2(\varphi(x * y))$$
$$= {\lambda_A}^2(\varphi(x) * \varphi(y))$$
$$\leq \max\{{\lambda_A}^2(\varphi(x)), {\lambda_A}^2(\varphi(y))\}$$
$$= \max\left\{\left(\varphi^{-1}({\lambda_A}^2)\right)(x), \left(\varphi^{-1}({\lambda_A}^2)\right)(y)\right\}$$

Hence $\varphi^{-1}(A)$ is an (4,2)-fuzzy sub algebra (of G_1) in τ_1 and consequently, φ is an (4,2)-fuzzy continuous function which maps (G_1, τ_1) to (G_2, τ_2).

Definition-3.6: Let (G_1, τ_1) and (G_2, τ_2) be (4,2)-fuzzy topology sub algebras. A function

$\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is said to be an (4,2)-fuzzy homomorphism if it satisfies the following conditions:

- φ is an injective and surjective function.
- φ is fuzzy continues function which maps G_1 to G_2.
- φ^{-1} is fuzzy continues function which maps G_2 to G_1.

Definition-3.7: Let τ be an (4,2)-fuzzy topology of BCC-algebra G. An (4,2)-fuzzy topology (G, τ) is an (4,2)- fuzzy Hausdorff space if and only if for any discrete (4,2)-fuzzy point $x_1, x_2 \in G$, there exits (m, n)- fuzzy topology $F_1 = \langle \delta^4_{F_1}, \lambda^2_{F_1} \rangle$ and $F_2 = \langle \delta^4_{F_2}, \lambda^2_{F_2} \rangle$ such that $\delta^4_{F_1}(x_1) = 1$, $\lambda^2_{F_1}(x_1) = 0$, $\delta^4_{F_2}(x_2) = 1$, $\lambda^2_{F_2}(x_2) = 0$ and $F_1 \cap F_2 = 0\sim$.

Theorem-3.8: Let τ_1 and τ_2 be (4,2)-fuzzy topologies on the BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an (4,2)-fuzzy homomorphism. Then G_1 is an (4,2)-fuzzy Hausdorff space if and only if G_2 is an (4,2)-fuzzy Hausdorff space.

Proof: Suppose that G_1 is a (4,2)-fuzzy Hausdorff space.

Let x_1, x_2 be the (4,2)-fuzzy point in τ_2 with $x \neq y$ where $x, y \in G_1$.

Then $\varphi^{-1}(x) \neq \varphi^{-1}(y)$ because φ is injective function.

For $z \in G_1$, $\left(\varphi^{-1}(x_1)\right)(z) = x_1\left(\varphi(z)\right)$

$$= \begin{cases} s \in [0,1], & \text{if } \varphi(z) = x \\ 0, & \text{if } \varphi(z) \neq x \end{cases} = \begin{cases} s \in [0,1], & \text{if } z = \varphi^{-1}(x) \\ 0, & \text{if } z \neq \varphi^{-1}(x) \end{cases}$$

$$= \left(\varphi^{-1}(x)\right)_1(z).$$

That is, $\left(\varphi^{-1}(x_1)\right)(z) = \left(\varphi^{-1}(x)\right)_1(z)$ for all $z \in G$. Hence $\varphi^{-1}(x_1)=\left(\varphi^{-1}(x)\right)_1$.

Similarly we can also prove that $\varphi^{-1}(x_2)=\left(\varphi^{-1}(x)\right)_2$. Now by the definition of an (4,2)-fuzzy Hausdorff space, there exist (4,2)-fuzzy order F_1 and F_2 of $\varphi^{-1}(x_1)$ and $\varphi^{-1}(x_2)$ respectively such that $F_1 \cap F_2 = 0\sim$. Since φ is an (4,2)-fuzzy continuous map from G_2 to G_1, there exist (4,2)-fuzzy orders $\varphi(F_1)$ and $\varphi(F_2)$ of x_1 and x_2 respectively such that $\varphi(F_1) \cap \varphi(F_2) = \varphi(F_1 \cap F_2) = \varphi(0\sim) = 0\sim$. This implies that G_2 is a (4,2)-fuzzy Hausdorff space.

Conversely, if (G_2, τ_2) is a (4,2)-fuzzy Hausdorff space, then by using a similar argument as above and by the fact that both φ and φ^{-1} are (4,2)-fuzzy continuous functions, we can easily prove that (G_1, τ_1) is an (4,2)-fuzzy Hausdorff space. Hence the proof.

Definition-3.9: Let τ be an (m, n)- fuzzy topology on a BCC-algebra G. Then (G, τ) is called an (4,2)-fuzzy C_5-disconnected space if there exists an (4,2)-fuzzy open and closed set F such that $F \neq 0\sim$ and $F \neq 1\sim$.

Theorem-3.10: Let τ_1 and τ_2 be the (4,2)-fuzzy topology sub algebras G_1 and G_2 respectively and let $\varphi: G_1 \to G_2$ be an (4,2)-fuzzy continuous surjective function. If (G_1, τ_1) is an (4,2)-fuzzy C_5-connected space then (G_2, τ_2) is also an (4,2)-fuzzy C_5-connected space.

Proof: Assume that (G_2, τ_2) is a (4,2)- fuzzy C_5-disconnected. Then there exist an (4,2)-fuzzy open and closed set F such that $F \neq 0\sim$ and $F \neq 1\sim$.

Since φ is an (m, n)- fuzzy continuous function $\varphi^{-1}(F)$ is both (4,2)-fuzzy open and (4,2)-fuzzy closed set. In this case $\varphi^{-1}(F) \neq 0\sim$ or $\varphi^{-1}(F) \neq 1\sim$.

Since, $F = \varphi(\varphi^{-1}(F)) = \varphi(0\sim) = 0\sim$ and $F = \varphi(\varphi^{-1}(F)) = \varphi(1\sim) = 1\sim$.

We see that these results contradict to our assumption.

Hence the space (G_2, τ_2) must be (4,2)-fuzzy C_5-connected space.

Definition-3.11: Let τ be an (4,2)-fuzzy topology on a BCC-algebra G. An (4,2)-fuzzy topology (G, τ) is called an (4,2)-fuzzy disconnected space if there exist (4,2)- fuzzy open sets $A \neq 0\sim$ and $B \neq 0\sim$ such that $A \cup B = 0\sim$. Naturally, we call the set (G, τ) an (m, n)- fuzzy connected if (G, τ) is not (4,2)-fuzzy disconnected.

Theorem-3.12: Let τ_1 and τ_2 be (m, n)- fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an (4,2)-fuzzy continuous and surjective function. If G_1 is an (4,2)-fuzzy connected space, then so is G_2.

Proof: Suppose that G_2 is an (m, n)- fuzzy disconnected, then there exists (4,2)-fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ in G_2 such that $C \cup D = 1\sim$ and $C \cap D = 0\sim$.

Since φ is (4,2)-fuzzy continuous function, $A = \varphi^{-1}(C)$ and $B = \varphi^{-1}(D)$ are (m, n)- fuzzy open sets in G_1.

Clearly, $C \neq 0\sim$ implies that $A = \varphi^{-1}(C) \neq 0\sim$, and $D \neq 0\sim$ implies that $B = \varphi^{-1}(D) \neq 0\sim$.

Now $C \cup D = 1\sim$.

$\Rightarrow \varphi^{-1}(C \cup D) = \varphi^{-1}(1\sim)$.

$\Rightarrow \varphi^{-1}(C) \cup \varphi^{-1}(D) = 1\sim$ implies $A \cup B = 1\sim$ and

$C \cap D = 0\sim \Rightarrow \varphi^{-1}(C \cap D) = \varphi^{-1}(0\sim)$

$\Rightarrow \varphi^{-1}(C) \cap \varphi^{-1}(D) = 0\sim$ implies $A \cap B = 0\sim$.

This clearly contradicts our hypothesis.

Hence G_2 is an (4,2)-fuzzy connected space.

Definition-3.13: An (m, n)- fuzzy topology space (G, τ) is said to be an (4,2)-fuzzy strongly connected, if there exists no non-zero (4,2)-fuzzy closed sets A and B in G such that $\delta_A^4 + \delta_B^4 \leq 1$ and $\lambda_A^2 + \lambda_B^2 \geq 1$.

The following fact follows immediately from the definition.

Propositon-3.14: G is (4,2)-fuzzy strongly connected if and only if there exist an (4,2)-fuzzy open sets A and B in G such that $A \neq 1\sim \neq B$ and $\delta_A^4 + \delta_B^4 \geq 1$, $\lambda_A^2 + \lambda_B^2 \leq 1$.

We now formulate the following theorem.

Theorem-3.15: Let τ_1 and τ_2 be (4,2)-fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an (4,2)- fuzzy continuous and surjective mapping. If G_1 is an (4,2)-fuzzy strongly connected, then so is G_2.

Proof: Suppose that G_2 is not an (4,2)-fuzzy strongly connected. Then there exists (4,2)-fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ so that $\delta_C^4 + \delta_D^4 \leq 1$ and $\lambda_C^2 + \lambda_D^2 \geq 1$. Since φ is an (4,2)-fuzzy continuous function, $\varphi^{-1}(C)$ and $\varphi^{-1}(D)$ are (4,2)-fuzzy closed sets in G_1. Now we can deduce the following equalities;

$$\delta^4_{\varphi^{-1}(C)} + \delta^4_{\varphi^{-1}(D)} = \varphi^{-1}(\delta_C^4) + \varphi^{-1}(\delta_D^4)$$

$$= \delta_C^4 \text{ o } \varphi + \delta_D^4 \text{ o } \varphi \leq 1 \text{ (Since } \delta_C^4 + \delta_D^4 \leq 1),$$

$$\lambda^2_{\varphi^{-1}(C)} + \lambda^2_{\varphi^{-1}(D)} = \varphi^{-1}(\lambda_C^2) + \varphi^{-1}(\lambda_D^2)$$

$$= \lambda_C^2 \text{ o } \varphi + \lambda_D^2 \text{ o } \varphi \geq 1 \text{ (Since } \lambda_C^2 + \lambda_D^2 \geq 1).$$

$\varphi^{-1}(C) \neq 0\sim$ and $\varphi^{-1}(D) \neq 0\sim$. This contradicts our hypothesis. Hence G_2 is an (4,2)-fuzzy strongly connected space.

Definition-3.16: Let τ be an (4,2)-fuzzy topology on a BCC-algebra G and A be an (4,2)-fuzzy BCC-algebra with (4,2)-fuzzy topology τ_A. Then A is called an (4,2)-fuzzy topological BCC-sub algebra if the self-mapping γ_a: $(A, \tau_A) \to (A, \tau_A)$ defined by $\gamma_a(x) = x * a$ for all $a \in G$, is a Relatively (4,2)-fuzzy continuous function.

Theorem-3.17: Let $\varphi: G_1 \to G_2$ be a homomorphism of BCC-algebras and τ and τ^* be (4,2)-fuzzy topologies on G_1 and G_2 respectively such that $\tau = \varphi^{-1}(\tau^*)$. If B is an (4,2)-fuzzy topological BCC-sub algebra in G_2, then $\varphi^{-1}(B)$ is an (4,2)-fuzzy topological BCC-sub algebra in G_1.

Theorem-3.18: Let $\varphi: G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively (4,2)-fuzzy topologies on the spaces G_1 and G_2 such that $\tau = \varphi^{-1}(\tau^*)$. If A is an (4,2)-fuzzy topological BCC-sub algebra in G_1, then $\varphi^{-1}(A)$ is an (4,2)-fuzzy topological BCC-sub algebra in G_2.

4. (4,2)-fuzzy topological BCC-ideals

Definition-4.1: An (4,2)-fuzzy set $A = \{\langle \delta_A, \lambda_A \rangle\}$ in a BCK-algebra G is called an (4,2)-fuzzy BCK-ideal of G if the following conditions are satisfied;

(i) $\delta_A^4(0) \geq \delta_A^4(x)$ and $\lambda_A^2(0) \leq \lambda_A^2(x)$,

(ii) $\delta_A^4(x) \geq \min\{\delta_A^4(x * y), \delta_A^4(y)\}$

(iii) $\lambda_A^2(x) \leq \max\{\lambda_A^2(x * y), \lambda_A^2(y)\}$ for all $x, y \in G$.

Definition-4.2: An (4,2)-fuzzy set $A = \langle \delta_A, \lambda_A \rangle$ in G is called an (4,2)-fuzzy BCC-ideal of G if it satisfies the following conditions;

(4,2) F_1: $\delta_A^4(0) \geq \delta_A^4(x)$ and $\lambda_A^2(0) \leq \lambda_A^2(x)$

(4,2) F_2: $\delta_A^4(x * z) \geq \min\{\delta_A^4((x * y) * z), \delta_A^4(y)\}$

(4,2) F_3: $\lambda_A^2(x * z) \leq \max\{\lambda_A^2((x * y) * z), \lambda_A^2(y)\}$ for all $x, y, z \in G$.

Putting $z = 0$ in (m, n) F_2 and (4,2) F_3, then we can easily see that an (4,2)-fuzzy BCC-ideal is an (m, n)- fuzzy BCK-ideal. However, the converse does not hold.

Example-4.3: Let $G = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra with the following Cayley table;

+	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $A = \langle \delta_A, \lambda_A \rangle$ be an (4,2)-fuzzy set in G defined by $\delta_A^4(5) = 0.02$, $\delta_A^4(x) = 0.4$, $\lambda_A^2(5) = 0.2$ and $\lambda_A^2(x) = 0.04$ for all $x \neq 5$, then A is an (4,2)-fuzzy BCC-ideal of a BCC-algebra G.

Theorem-4.4: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an (4,2)- fuzzy BCC-ideal of G_2. Then $\varphi^{-1}(B)$ is an (4,2)-fuzzy BCC-ideal of G_1.

Proof: It can be easily seen that

$\delta_{\varphi^{-1}(B)}^4(0) \geq \delta_{\varphi^{-1}(B)}^4(x)$ and $\lambda_{\varphi^{-1}(B)}^2(0) \leq \lambda_{\varphi^{-1}(B)}^2(x)$, for all $x \in G_1$.

For any $x, y, z \in G_1$, we can deduce the following

$$\begin{aligned}
\delta_{\varphi^{-1}(B)}^4(x * z) &= \delta_B^4(\varphi(x * z)) \\
&\geq \min\left\{\delta_B^4\left(\varphi((x * y) * z)\right), \delta_B^4(\varphi(y))\right\} \\
&= \min\left\{\delta_B^4\left((\varphi(x) * \varphi(y)) * \varphi(z)\right), \delta_B^4(\varphi(y))\right\} \\
&= \min\left\{\delta_{\varphi^{-1}(B)}^4((x * y) * z), \delta_{\varphi^{-1}(B)}^4(y)\right\}.
\end{aligned}$$

Also

$$\lambda^2_{\varphi^{-1}(B)}(x*z) = \lambda^2_B(\varphi(x*z))$$
$$\leq \max\left\{\lambda^2_B\left(\varphi\big((x*y)*z\big)\right), \lambda^2_B(\varphi(y))\right\}$$
$$= \max\left\{\lambda^2_B\left(\big(\varphi(x)*\varphi(y)\big)*\varphi(z)\right), \lambda^2_B(\varphi(y))\right\}$$
$$= \max\left\{\lambda^2_{\varphi^{-1}(B)}\big((x*y)*z\big), \lambda^2_{\varphi^{-1}(B)}(y)\right\}$$

Hence $\varphi^{-1}(B)$ is an (4,2)-fuzzy BCC-ideal of G_1.

Corollarly-4.5: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an (4,2)-fuzzy BCK-ideal of G_2. Then $\varphi^{-1}(B)$ is an (4,2)-fuzzy BCK-ideal of G_1.

Since an (m, n)- fuzzy BCC-ideal / BCK-ideal is an (4,2)- fuzzy sub algebra, as a consequence of the above results and theorem-3.17, we obtain the following corollary.

Corollarly-4.6: Let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be a homomorphism of the BCC-algebras. Let τ_1 and τ_2 be the (4,2)-fuzzy topologies on G_1 and G_2 respectively such that $\tau_2 = \varphi^{-1}(\tau_1)$. If B is (4,2)-fuzzy topological BCC-ideal / BCK-ideal of G_2 with the membership function δ^4_B, then $\varphi^{-1}(B)$ is a (4,2)-fuzzy topological BCC-ideal / BCK-ideal of G_1 with the membership function $\delta^m_{\varphi^{-1}(B)}$.

Theorem-4.7: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If A is an (4,2)-fuzzy BCC-ideal of G_1, then the homomorphic image $\varphi(A)$ of A is still an (4,2)-fuzzy BCC-ideal of G_2.

Proof: Let A be an (4,2)-fuzzy topological BCC-ideal of G_1. Then, it is trivial that

$\delta^4_{\varphi(A)}(0) \geq \delta^4_{\varphi(A)}(x)$ and $\lambda^2_{\varphi(A)}(0) \leq \lambda^2_{\varphi(A)}(x)$, for all $x \in G_2$.

Take $x, y, z \in G_2$, and let $x_0 \in \varphi^{-1}(x)$, $y_0 \in \varphi^{-1}(y)$, $z_0 \in \varphi^{-1}(z)$ such that

$\delta^4_A(x_0) = \sup_{t\in \varphi^{-1}(x)} t$, $\delta^4_A(y_0) = \sup_{t\in \varphi^{-1}(y)} t$ and $\delta^4_A(z_0) = \sup_{t\in \varphi^{-1}(z)} t$.

Then we can deduce the following,

$$\delta^4_{\varphi(A)}(x*z) = \sup_{t\in \varphi^{-1}(x*z)}\left(\delta^4_A(t)\right)$$
$$\geq \delta^4_A(x_0*z_0)$$
$$\geq \min\{\delta^4_A\big((x_0*y_0)*z_0\big), \delta^4_A(y_0)\}$$
$$= \min\left\{\sup_{t\in \varphi^{-1}((x*y)*z)}\left(\delta^4_A(t)\right), \sup_{t\in \varphi^{-1}(y)}\left(\delta^4_A(t)\right)\right\}$$
$$= \min\{\delta^4_{\varphi(A)}\big((x*y)*z\big), \delta^4_{\varphi(A)}(y)\}$$

and $\quad \lambda^2_{\varphi(A)}(x*z) = \inf_{t\in \varphi^{-1}(x*z)}\left(\lambda^2_{\varphi(A)}(t)\right) \leq \lambda^2_A(x_0*z_0)$

$$\leq \max\{\lambda_A^2((x_0 * y_0) * z_0), \lambda_A^2(y_0)\}$$

$$= \max\left\{\inf_{t\in \varphi^{-1}((x*y)*z)}\left(\lambda_A^2(t)\right), \inf_{t\in \varphi^{-1}(y)}\left(\lambda_A^2(t)\right)\right\}$$

$$= \max\{\lambda_{\varphi(A)}^2((x * y) * z), \lambda_{\varphi(A)}^2(y)\}$$

Hence $\varphi(A) = \langle\varphi_{sup}(\delta_A), \varphi_{inf}(\lambda_A)\rangle$ is induced an (4,2)-fuzzy BCC-ideal of G_2.

Putting $z = 0$ in the above theorem, we obtain:

Corollarly-4.8: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If If A is an (4,2)-fuzzy BCK-ideal of G_1, then the homomorphic image $\varphi(A)$ of A is still an (4,2)-fuzzy BCK-ideal of G_2.

Summing up theorem-3.18, theorem-4.7 and corollary-4.8, we conclude the following theorem.

Theorem-4.9: Let $\varphi: G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively(4,2)-fuzzy topologies on the spaces G_1 and G_2 such that $\varphi(\tau) = \tau^*$. If A is an (4,2)-fuzzy topological BCC-ideal / BCK-ideal in G_1, then $\varphi(A)$ is also an (4,2)-fuzzy topological BCC-ideal / BCK-ideal in G_2.

Conclusion: we study the concept of (4,2)-fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also discussed the characteristic of the homomorphic image and inverse image of (4,2)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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